Problem 1.31

Solve the following differential equations:

(*i*) $-xy' + y = xy^2$ $[y(1) = 1]$;

Solution

This is a Bernoulli equation. First get it into standard form by dividing both sides by $-x$.

$$y' - \frac{1}{x}y = -y^2$$

Divide both sides now by y^2.

$$y^{-2}y' - \frac{1}{x}y^{-1} = -1$$

Make the substitution,

$$u = y^{-1}$$
$$\frac{du}{dx} = -y^{-2}\frac{dy}{dx} \quad \rightarrow \quad -\frac{du}{dx} = y^{-2}\frac{dy}{dx}$$

Plug these expressions into the ODE.

$$-\frac{du}{dx} - \frac{1}{x}u = -1$$

Multiply both sides by -1.

$$\frac{du}{dx} + \frac{1}{x}u = 1$$

This is a first-order inhomogeneous equation that can be solved by multiplying both sides by an integrating factor I.

$$I = e^{\int^x \frac{1}{s}\,ds} = e^{\ln x} = x$$

Proceed with the multiplication.

$$x\frac{du}{dx} + u = x$$

The left side is now exact and can be written as $d/dx(Iu)$ as a result of the product rule.

$$\frac{d}{dx}(xu) = x$$

Integrate both sides of the equations with respect to x.

$$xu = \frac{1}{2}x^2 + C$$

Divide both sides by x to solve for u.

$$u(x) = \frac{1}{2}x + \frac{C}{x}$$

Now that the integration is done, change back to the original variable y.

$$\frac{1}{y} = \frac{1}{2}x + \frac{C}{x}$$

Write the right side as one term by combining the fractions.

$$\frac{1}{y} = \frac{x^2 + 2C}{2x}$$

Invert both sides to solve for y.

$$y(x) = \frac{2x}{x^2 + 2C}$$

Now that we have the general solution we can apply the initial condition to determine the constant in the denominator.

$$y(1) = \frac{2}{1 + 2C} = 1$$

Solving this equation yields $C = 1/2$. Therefore,

$$y(x) = \frac{2x}{x^2 + 1}.$$

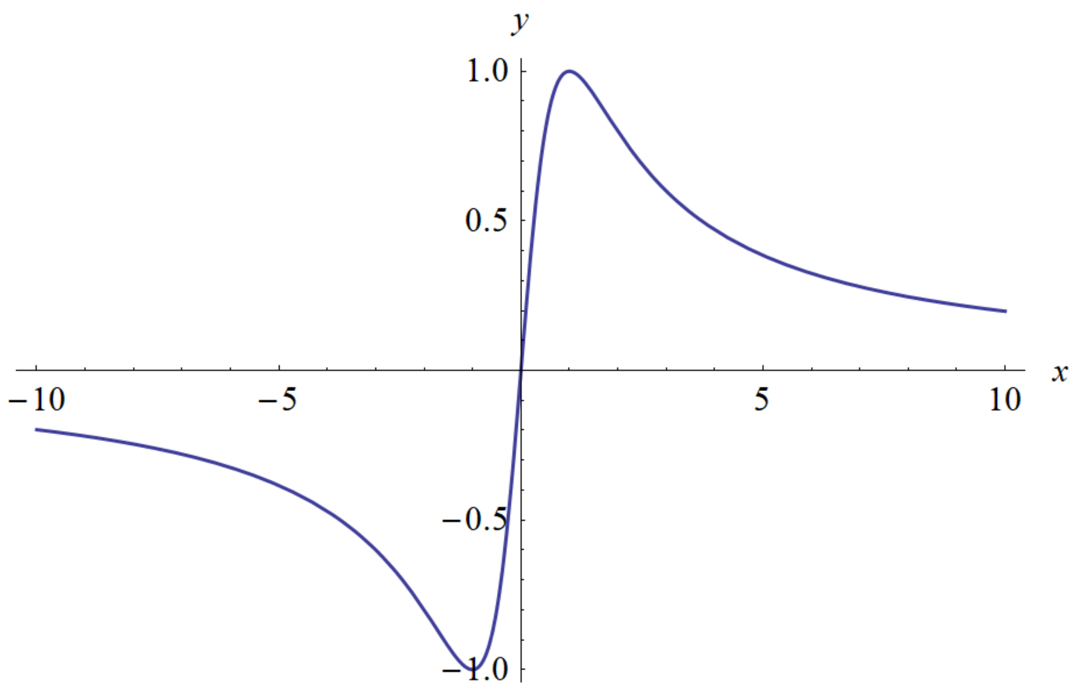

Figure 1: Plot of the solution for $-10 < x < 10$.