Problem 1.31

Solve the following differential equations:

(*j*) $y'' - (1+x)^{-2}(y')^2 = 0$ $[y(0) = y'(0) = 1]$;

Solution

This ODE is first-order in y', so make the substitution,

$$u = y'$$
$$u' = y''.$$

Plugging these expressions into the ODE yields

$$u' - \frac{1}{(1+x)^2}u^2 = 0,$$

which can be solved by separation of variables. Bring the second term over to the right.

$$\frac{du}{dx} = \frac{1}{(1+x)^2}u^2$$

Separate variables.

$$\frac{du}{u^2} = \frac{dx}{(1+x)^2}$$

Integrate both sides.

$$-\frac{1}{u} = -\frac{1}{1+x} + C$$

Multiply both sides by -1 and combine the two terms on the right into one.

$$\frac{1}{u} = \frac{1 - C(1+x)}{1+x}$$

Invert both sides now to solve for u.

$$u(x) = \frac{1+x}{1 - C(1+x)}$$

Now that the integration is done, change back to the original variable y.

$$y' = \frac{1+x}{1 - C(1+x)}$$

At this point we can apply the first initial condition, $y'(0) = 1$, to determine C.

$$y'(0) = \frac{1}{1-C} = 1$$

Solving for C gives $C = 0$. So we have

$$y' = 1 + x.$$

Integrate both sides with respect to x to solve for y.

$$y(x) = x + \frac{1}{2}x^2 + D$$

Use the second initial condition, $y(0) = 1$, to determine D.

$$y(0) = D = 1$$

Therefore,

$$y(x) = x + \frac{1}{2}x^2 + 1.$$

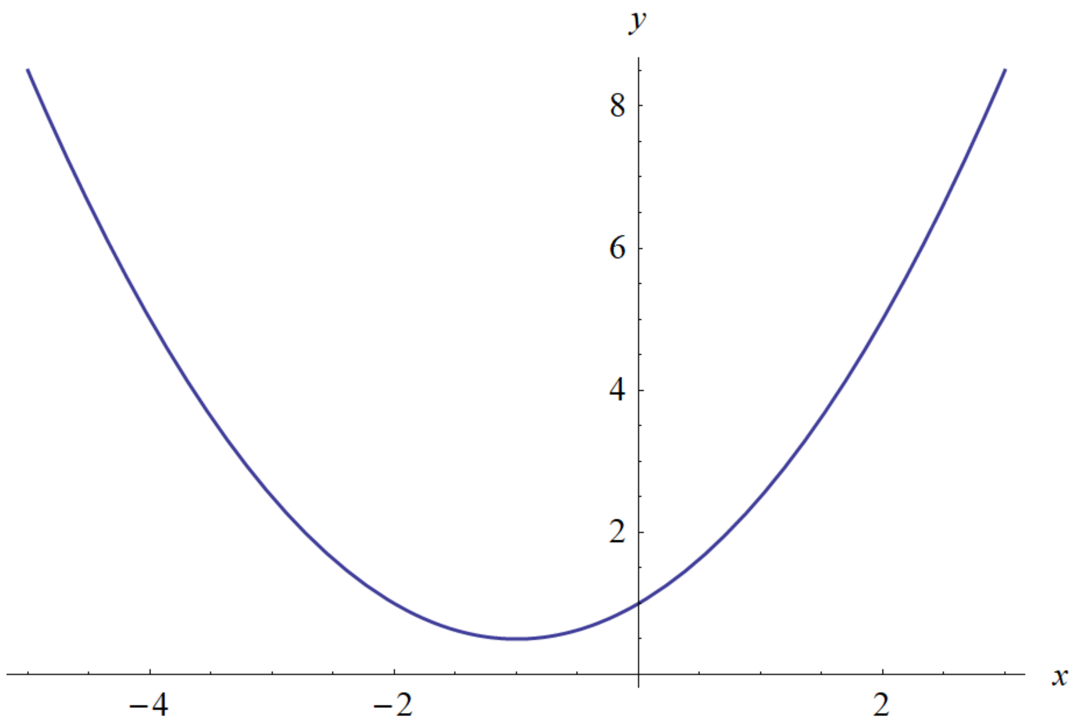

Figure 1: Plot of the solution for $-5 < x < 3$.