Problem 1.31

Solve the following differential equations:

(l) $y'' = (y')^2 e^{-y}$ (if $y' = 1$ at $y = \infty$, find y' at $y = 0$);

Solution

Divide both sides by y'.

$$\frac{y''}{y'} = y' e^{-y}$$

Rewrite the left side as follows.

$$\frac{d}{dx} \ln y' = y' e^{-y}$$

Rewrite the right side as follows.

$$\frac{d}{dx} \ln y' = \frac{d}{dx}(-e^{-y})$$

Integrate both sides with respect to x.

$$\ln y' = -e^{-y} + C.$$

Exponentiate both sides.

$$y' = e^C e^{-e^{-y}}$$

Use a new arbitrary constant A.

$$y' = A e^{-e^{-y}}$$

Now that we solved for y' in terms of y, we can use the provided boundary condition to determine A. As $y \to \infty$, $e^{-y} \to 0$, so we have

$$\lim_{y \to \infty} y' = A e^0 = A = 1.$$

Now that we know A, we can find y' when $y = 0$.

$$\lim_{y \to 0} y' = e^{-e^0}$$

Therefore, y' at $y = 0$ is equal to e^{-1}.