Problem 1.31

Solve the following differential equations:

(m) $y' = |y - x|$ [if $y(0) = \frac{1}{2}$, find $y(1)$];

Solution

The right side prompts the substitution,

$$u = y - x$$
$$\frac{du}{dx} = \frac{dy}{dx} - 1 \quad \rightarrow \quad \frac{du}{dx} + 1 = \frac{dy}{dx}.$$

Plug these expressions into the ODE.

$$\frac{du}{dx} + 1 = |u|$$

Bring 1 to the right side.

$$\frac{du}{dx} = |u| - 1$$

The absolute value is defined as

$$\begin{cases} u & u > 0 \\ -u & u < 0, \end{cases}$$

so there are two cases to consider here.

<u>Case I: $u > 0$</u>

Here we consider the first case.

$$\frac{du}{dx} = u - 1$$

This equation can be solved with separation of variables.

$$\frac{du}{u - 1} = dx$$

Integrate both sides.

$$\ln|u - 1| = x + C$$

Exponentiate both sides.

$$|u - 1| = e^x e^C$$

Eliminate the absolute value sign by introducing $\pm$ on the right side.

$$u - 1 = \pm e^C e^x$$

Use a new arbitrary constant.

$$u - 1 = Ae^x$$

Bring 1 to the right side to solve for u.

$$u(x) = 1 + Ae^x, \quad u > 0$$

Change back now to the original variable y.

$$y - x = 1 + Ae^x$$

Thus, for the first case we have

$$y(x) = x + 1 + Ae^x, \quad y - x > 0.$$

Case II: $u < 0$

Here we consider the second case.

$$\frac{du}{dx} = -u - 1$$

This equation can be solved with separation of variables.

$$\frac{du}{u+1} = -dx$$

Integrate both sides.

$$\ln|u + 1| = -x + C$$

Exponentiate both sides.

$$|u + 1| = e^{-x}e^C$$

Eliminate the absolute value sign by introducing $\pm$ on the right side.

$$u + 1 = \pm e^C e^{-x}$$

Use a new arbitrary constant.

$$u + 1 = Be^{-x}$$

Bring 1 to the right side to solve for u.

$$u(x) = -1 + Be^{-x}, \quad u < 0$$

Change back now to the original variable y.

$$y - x = -1 + Be^{-x}$$

Thus, for the second case we have

$$y(x) = x - 1 + Be^{-x}, \quad y - x < 0.$$

Putting the results of these two cases together, we have for the general solution

$$y(x) = \begin{cases} x + 1 + Ae^x & y - x > 0 \\ x - 1 + Be^{-x} & y - x < 0 \end{cases}.$$

To determine one of the constants, we use the provided initial condition, $y(0) = \frac{1}{2}$. Since y is bigger than x, we apply it to the first case.

$$y(0) = 1 + A = \frac{1}{2} \quad \rightarrow \quad A = -\frac{1}{2}$$

The solution is now

$$y(x) = \begin{cases} x + 1 - \frac{1}{2}e^x & y - x > 0 \\ x - 1 + Be^{-x} & y - x < 0 \end{cases}.$$

To determine the second unknown constant, we require that the solution be continuous everywhere, that is, when $y - x = 0$, the two expressions for $y(x)$ must yield the same result. Bring x to the left side.

$$y - x = \begin{cases} 1 - \dfrac{1}{2}e^x = 0 \\ -1 + Be^{-x} = 0 \end{cases}$$

We have here a system of two equations for two unknowns, x and B. Solving the system gives us $x = \ln 2$ and $B = 2$. Therefore, the solution to the ODE is

$$y(x) = \begin{cases} x + 1 - \frac{1}{2}e^x & y - x > 0 \\ x - 1 + 2e^{-x} & y - x < 0 \end{cases}.$$

Although we have determined the constants, this equation is only implicit for $y(x)$. Our aim now is to write an explicit expression for y, that is, one that depends only on x. The interpretation of this solution is as follows: above the line $y = x$, we use the first expression for $y(x)$ and below the same line, we use the second expression for $y(x)$. What we have to do is graph the functions and find out for what values of x this occurs.

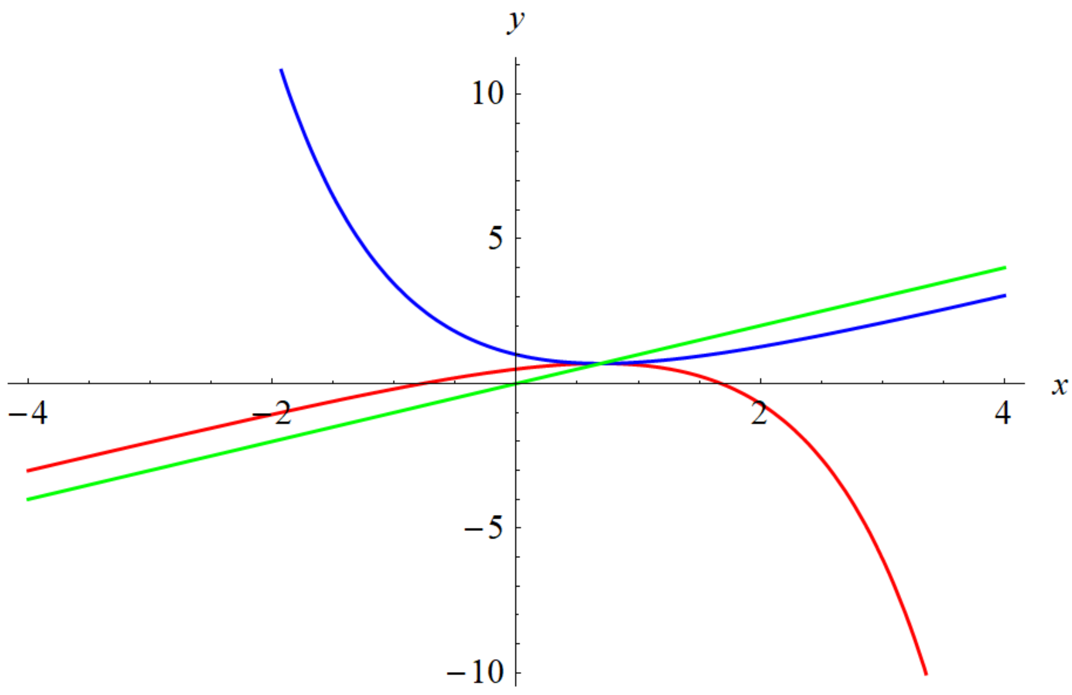

Figure 1: This is a plot of three functions for $-4 < x < 4$. The first expression for $y(x)$ is in red, the second expression for $y(x)$ is in blue, and the line, $y = x$, is in green.

As can be seen from the graph, the red line is above the green line to the left of the point of intersection, $x = \ln 2$. Also, the blue line is below the green line to the right of $x = \ln 2$. Therefore, the explicit solution for $y(x)$ is this.

$$y(x) = \begin{cases} x + 1 - \frac{1}{2}e^x & x < \ln 2 \\ x - 1 + 2e^{-x} & x \geq \ln 2 \end{cases}$$

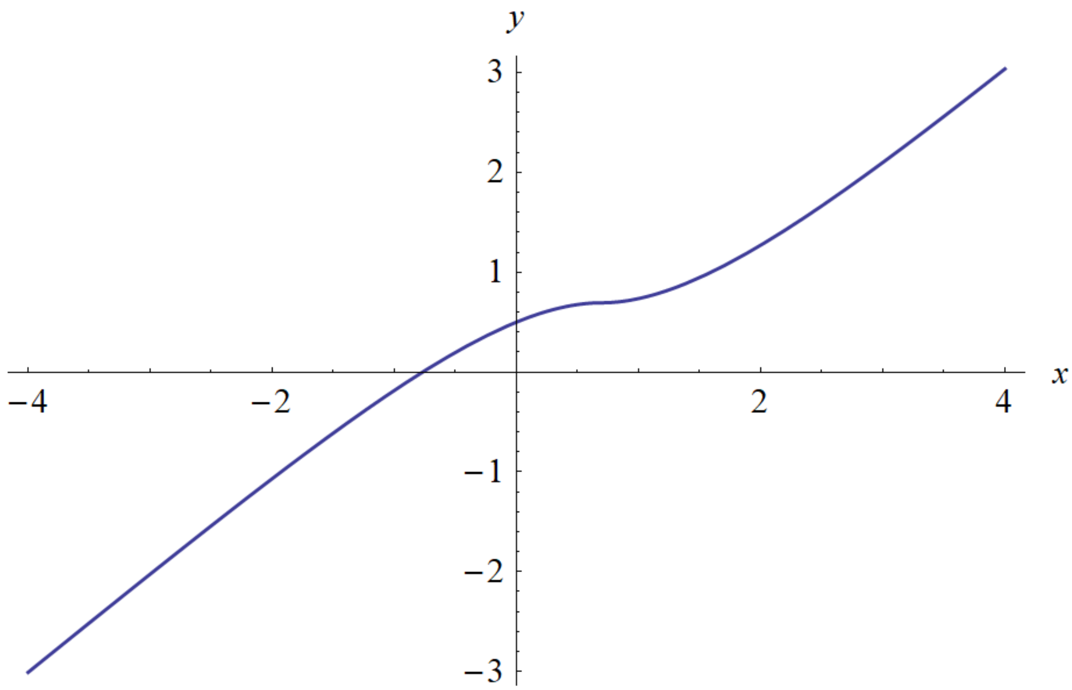

Figure 2: Plot of the solution for $-4 < x < 4$.

Finally, we are in a position to answer the question. Since $\ln 2 \approx 0.69$, we use the second expression to determine $y(1)$.

$$y(1) = 2e^{-1} \approx 0.73$$