Problem 1.31

Solve the following differential equations:

(*o*) $y' = (x^4 - 3x^2y^2 - y^3)/(2x^3y + 3y^2x)$;

Solution

Bring all terms over to the left side.

$$y^3 + 3x^2y^2 - x^4 + (2x^3y + 3y^2x)\frac{dy}{dx} = 0$$

This ODE is of the form,

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0.$$

Check to see whether $M_y = N_x$ or not. It it's not, then we'll have to multiply both sides by an integrating factor.

$$\frac{\partial M}{\partial y} = 3y^2 + 6x^2y$$
$$\frac{\partial N}{\partial x} = 6x^2y + 3y^2$$

$M_y = N_x$, so the ODE is exact. This implies that there exists a potential function $\phi(x,y)$ such that

$$\frac{\partial \phi}{\partial x} = M(x,y) \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N(x,y). \tag{2}$$

The ODE thus becomes

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0.$$

Recall that the differential of a function of two variables $\phi(x,y)$ is

$$d\phi = \frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy.$$

Divide both sides by dx to obtain the relationship between the total derivative of $\phi(x,y)$ and the partial derivatives of $\phi(x,y)$.

$$\frac{d\phi}{dx} = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx}$$

Consequently, the ODE becomes

$$\frac{d\phi}{dx} = 0.$$

Integrate both sides with respect to x to obtain the solution to the ODE.

$$\phi(x,y) = A,$$

where A is an arbitrary constant. Our aim now is to determine $\phi(x, y)$ using equations (1) and (2).

$$\frac{\partial \phi}{\partial x} = y^3 + 3x^2y^2 - x^4 \tag{1}$$

$$\frac{\partial \phi}{\partial y} = 2x^3y + 3y^2x \tag{2}$$

Integrate the second equation partially with respect to y to solve for ϕ. Note that we could integrate the first equation partially with respect to x to solve for ϕ as well. We would get the same answer either way.

$$\begin{aligned}
\phi(x, y) &= \int^y \left.\frac{\partial \phi}{\partial y}\right|_{y=s} ds + f(x) \\
&= \int^y (2x^3s + 3s^2x)\, ds + f(x) \\
&= \int^y 2x^3s\, ds + \int^y 3s^2x\, ds + f(x) \\
&= 2x^3 \int^y s\, ds + 3x \int^y s^2\, ds + f(x) \\
&= x^3y^2 + xy^3 + f(x)
\end{aligned}$$

In order to determine the arbitrary function $f(x)$, we have to use equation (1). Differentiate the expression we just obtained with respect to x.

$$\frac{\partial \phi}{\partial x} = 3x^2y^2 + y^3 + f'(x)$$

Comparing this with equation (1), we see that $f'(x)$ has to be equal to $-x^4$ in order to be consistent. Hence, $f(x) = -x^5/5$. Therefore, the general solution to the ODE is the following.

$$x^3y^2 + xy^3 - \frac{x^5}{5} = A$$