Problem 1.31

Solve the following differential equations:

(*p*) $(x^2 + y^2)y' = xy, \; y(e) = e;$

Solution

Divide both sides by $x^2 + y^2$ to solve for y'.

$$y' = \frac{xy}{x^2 + y^2}$$

Multiply the numerator and denominator on the right side by $1/x^2$.

$$y' = \frac{xy}{x^2 + y^2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \frac{\frac{y}{x}}{1 + \frac{y^2}{x^2}} = \frac{\frac{y}{x}}{1 + \left(\frac{y}{x}\right)^2}$$

The right-hand side suggests the substitution,

$$u = \frac{y}{x} \quad \to \quad xu = y$$
$$u + x\frac{du}{dx} = \frac{dy}{dx}.$$

The ODE is transformed to

$$u + x\frac{du}{dx} = \frac{u}{1 + u^2}.$$

Bring u to the right side.

$$x\frac{du}{dx} = -\frac{u^3}{1 + u^2}$$

This ODE can be solved by separation of variables.

$$\frac{1 + u^2}{u^3}\,du = -\frac{dx}{x}$$

Integrate both sides.

$$\int (u^{-3} + u^{-1})\,du = -\ln|x| + C$$

$$\frac{1}{-2}u^{-2} + \ln|u| = -\ln|x| + C$$

Bring $\ln|x|$ to the left and combine it with $\ln|u|$.

$$-\frac{1}{2}\frac{1}{u^2} + \ln|xu| = C$$

Now that the integration is done, change back to the original variable y.

$$-\frac{1}{2}\frac{x^2}{y^2} + \ln|y| = C$$

Multiply both sides by -2.

$$\frac{x^2}{y^2} - 2\ln y = -2C$$

We can determine $-2C$ by using the provided boundary condition, $y(e) = e$.

$$1 - 2\ln e = -2C \quad \rightarrow \quad -2C = -1$$

Therefore,

$$\frac{x^2}{y^2} - 2\ln y = -1.$$

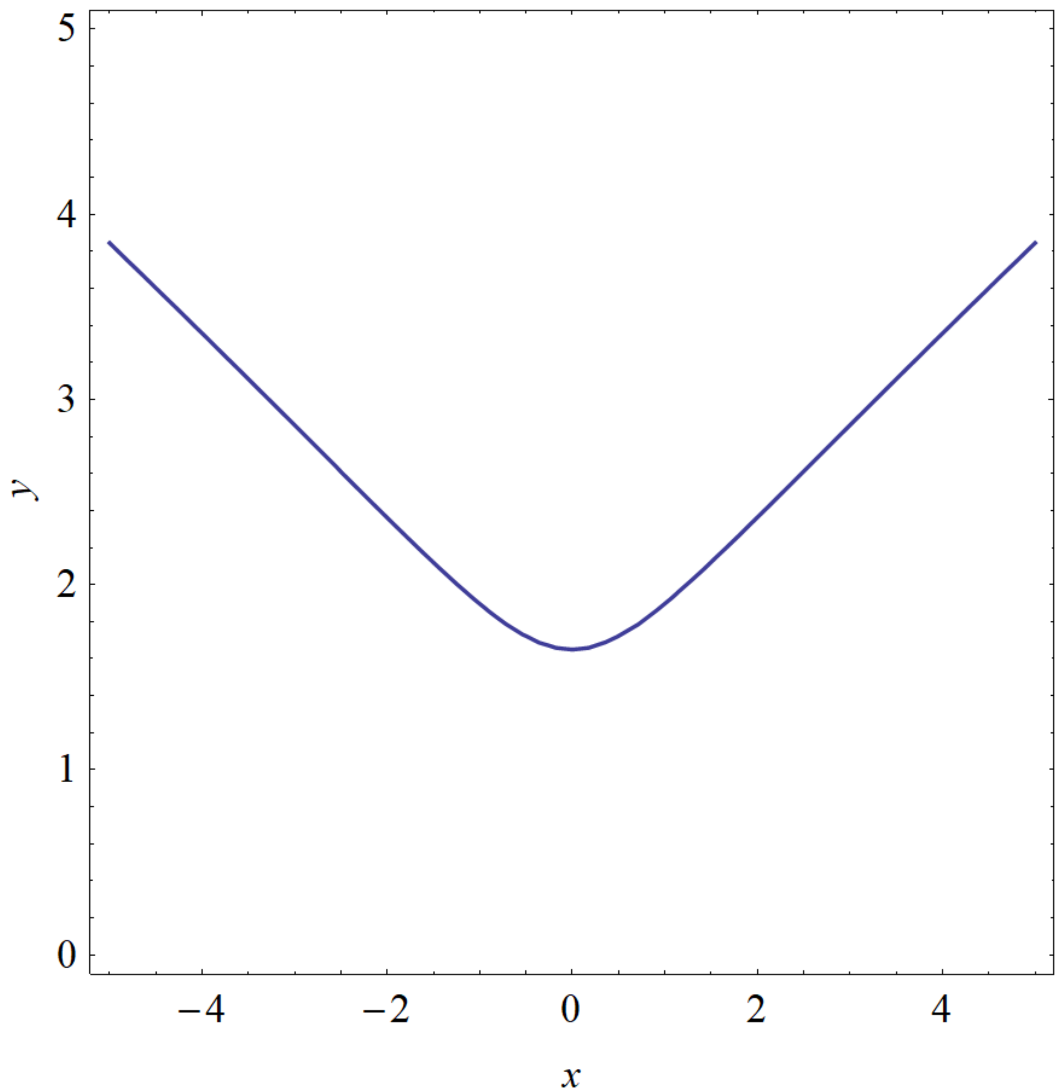

Figure 1: Plot of the solution for $-5 < x < 5$ and $0 < y < 5$.