Problem 1.31

Solve the following differential equations:

(t) $xy' = y + \sqrt{xy}$;

Solution

Divide both sides by x.

$$y' = \frac{y}{x} + \frac{1}{x}\sqrt{xy}$$

Bring x inside the square root.

$$y' = \frac{y}{x} + \sqrt{\frac{y}{x}}$$

The right side prompts the substitution,

$$u = \frac{y}{x} \quad \rightarrow \quad xu = y$$
$$u + x\frac{du}{dx} = \frac{dy}{dx},$$

Plugging these expressions into the ODE gives us

$$u + x\frac{du}{dx} = u + \sqrt{u}.$$

Cancelling u, we have here an ODE we can solve with separation of variables.

$$x\frac{du}{dx} = \sqrt{u}$$

Separate variables.

$$u^{-1/2}\,du = \frac{dx}{x}$$

Integrate both sides. Use $\ln C$ for the integration constant.

$$2u^{1/2} = \ln|x| + \ln C$$

Combine the logarithms.

$$2u^{1/2} = \ln C|x|$$

Because C is arbitrary, we can drop the absolute value sign. Divide both sides by 2.

$$u^{1/2} = \frac{1}{2}\ln Cx$$

Square both sides to solve for u.

$$u(x) = \frac{1}{4}(\ln Cx)^2$$

Change back now to the original variable y.

$$\frac{y}{x} = \frac{1}{4}(\ln Cx)^2$$

Multiply both sides by x to solve for y. Therefore,

$$y(x) = \frac{x}{4}(\ln Cx)^2.$$