Problem 1.31

Solve the following differential equations:

(u) $(xy)y' + y\ln y = 2xy$ [try an integrating factor of the form $I = I(y)$]; [TYPO: The first term should be xy']

Solution

In order for an integrating factor of the form $I = I(y)$ to work, the ODE has to be

$$xy' + y\ln\, y = 2xy.$$

I confirmed this with one of the authors, Mr. Bender.

Solution by an Integrating Factor

Bring $2xy$ to the left side and factor y.

$$xy' + y(\ln\, y - 2x) = 0$$

Multiply both sides by the integrating factor $I(y)$.

$$xI(y)y' + yI(y)(\ln\, y - 2x) = 0$$

For this ODE to be exact, we require that

$$\frac{\partial}{\partial y}[yI(y)(\ln\, y - 2x)] = \frac{\partial}{\partial x}[xI(y)].$$

The right side is a function of y only. For the left side to be as well, $yI(y)$ must be equal to a constant. An appropriate integrating factor is thus

$$I(y) = \frac{1}{y}.$$

The ODE becomes

$$\frac{x}{y}y' + \ln\, y - 2x = 0,$$

which is exact. This means there exists a potential function $\phi = \phi(x, y)$ such that

$$\frac{\partial\phi}{\partial x} = \ln\, y - 2x \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \frac{x}{y}. \tag{2}$$

Substituting these into the ODE, we get

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0.$$

The differential of a function $\phi(x, y)$ is defined as

$$d\phi = \frac{\partial\phi}{\partial x}\,dx + \frac{\partial\phi}{\partial y}\,dy.$$

Dividing both sides by dx gives the relationship between the total derivative of ϕ and the partial derivatives of ϕ.

$$\frac{d\phi}{dx} = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx}$$

Hence, the ODE simplifies to

$$\frac{d\phi}{dx} = 0.$$

Integrate both sides with respect to x to obtain the general solution.

$$\phi = A,$$

where A is an arbitrary constant. Our task now is to determine the potential function ϕ from equations (1) and (2). Integrate equation (1) partially with respect to x.

$$\begin{aligned} \phi(x, y) &= \int^x \left.\frac{\partial \phi}{\partial x}\right|_{x=s} ds + f(y) \\ &= \int^x (\ln\, y - 2s)\, ds + f(y) \\ &= x \ln\, y - x^2 + f(y) \end{aligned}$$

To determine the arbitrary function $f(y)$, differentiate this expression partially with respect to y and compare it with equation (2).

$$\frac{\partial \phi}{\partial y} = \frac{x}{y} + f'(y)$$

We see that $f'(y)$ has to equal zero, which means $f(y) = B$, a constant. The potential function is consequently

$$\phi(x, y) = x \ln\, y - x^2 + B,$$

which means the general solution to the ODE is

$$x \ln\, y - x^2 + B = A.$$

Subtract B from both sides and use a new arbitrary constant C.

$$x \ln\, y - x^2 = C.$$

This equation can be solved for y explicitly. Bring x^2 to the right side.

$$x \ln y = C + x^2$$

Divide both sides by x.

$$\ln y = x + \frac{C}{x}$$

Exponentiate both sides to solve for y. Therefore,

$$y(x) = e^{x + C/x}.$$

Solution by a Substitution

$$xy' + y \ln y = 2xy$$

Divide both sides of the ODE by xy.

$$y^{-1}y' + \frac{1}{x}\ln y = 2$$

Make the substitution,

$$u = \ln y.$$

Take the derivative of both sides with respect to x to find out what y' is in terms of the new variable.

$$\frac{du}{dx} = y^{-1}\frac{dy}{dx}$$

Plug these expressions into the ODE.

$$\frac{du}{dx} + \frac{1}{x}u = 2$$

This is a first-order inhomogeneous equation that we can solve with an integrating factor I.

$$I = e^{\int^x \frac{1}{s}\,ds} = e^{\ln x} = x$$

Multiply both sides of the ODE by the integrating factor.

$$x\frac{du}{dx} + u = 2x$$

The left side can now be written as $d/dx(Iu)$ as a result of the product rule.

$$\frac{d}{dx}(xu) = 2x$$

Integrate both sides with respect to x.

$$xu = x^2 + C$$

Divide both sides by x to solve for u.

$$u(x) = x + \frac{C}{x}$$

Now that we have u, change back to the original variable y.

$$\ln y = x + \frac{C}{x}$$

Exponentiate both sides to solve for y. Therefore,

$$y(x) = e^{x+C/x}.$$