Problem 1.31

Solve the following differential equations:

(*v*) $(x \sin y + e^y)y' = \cos y;$

Solution

Bring $\cos y$ to the left side.

$$-\cos y + (x \sin y + e^y)y' = 0$$

This ODE has the form,

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0.$$

Check to see whether $M_y = N_x$. If it's not, we'll have to use an integrating factor.

$$\frac{\partial M}{\partial y} = \sin y$$
$$\frac{\partial N}{\partial x} = \sin y$$

The two partial derivatives are equal, which means the ODE is exact. This implies that there exists a potential function $\phi(x,y)$ such that

$$\frac{\partial \phi}{\partial x} = M(x,y) \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N(x,y). \tag{2}$$

Substituting these relations into the ODE gives

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0.$$

Recall that the differential of a function of two variabes, $\phi = \phi(x,y)$, is this.

$$d\phi = \frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy$$

Dividing both sides by dx gives us the relationship between the total derivative of ϕ and the partial derivatives of it.

$$\frac{d\phi}{dx} = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx}$$

Substitution into the ODE reduces it to

$$\frac{d\phi}{dx} = 0.$$

Integrating both sides with respect to x gives the general solution.

$$\phi = A,$$

where A is an arbitrary constant. Our task now is to find this potential function $\phi(x,y)$ using equations (1) and (2).

$$\frac{\partial \phi}{\partial x} = -\cos y \tag{1}$$
$$\frac{\partial \phi}{\partial y} = x \sin y + e^y \tag{2}$$

We will solve for ϕ by integrating both sides of equation (1) partially with respect to x. Note that we would get the same answer for ϕ integrating both sides of equation (2) partially with respect to y.

$$\begin{aligned} \phi(x,y) &= \int^x \left.\frac{\partial \phi}{\partial x}\right|_{x=s} ds + f(y) \\ &= \int^x -\cos y \, ds + f(y) \\ &= -x\cos y + f(y) \end{aligned}$$

Differentiate this expression we just obtained with respect to y.

$$\frac{\partial \phi}{\partial y} = x \sin y + f'(y)$$

Comparing this result with equation (2), we see that $f'(y)$ has to be equal to e^y in order to be consistent, which means $f(y) = e^y + C$. We thus have

$$-x\cos y + e^y + C = A$$

for the general solution to the ODE. Bring C to the left and use a new arbitrary constant B. Therefore,

$$-x\cos y + e^y = B$$

is the general (albeit implicit) solution for $y(x)$.