Problem 1.31

Solve the following differential equations:

(z) $xy' + y = y^2x^4$.

Solution

This is a Bernoulli equation. Start off by getting rid of the term multiplying y'. Divide both sides of the equation by x.

$$y' + \frac{1}{x}y = y^2x^3$$

Now divide both sides by y^2.

$$y^{-2}y' + \frac{1}{x}y^{-1} = x^3$$

Make the substitution,

$$u = y^{-1}$$
$$\frac{du}{dx} = (-1)y^{-2}\frac{dy}{dx} \quad \rightarrow \quad -\frac{du}{dx} = y^{-2}\frac{dy}{dx}.$$

Plug these expressions into the ODE.

$$-\frac{du}{dx} + \frac{1}{x}u = x^3$$

This is a first-order ODE that can be solved with an integrating factor. Multiply both sides by -1.

$$\frac{du}{dx} - \frac{1}{x}u = -x^3$$

The integrating factor is this.

$$I = e^{\int^x -\frac{1}{s}\,ds} = e^{-\ln x} = x^{-1}$$

Multiply both sides by I.

$$\frac{1}{x}\frac{du}{dx} - \frac{1}{x^2}u = -x^2$$

The left side is now exact and can be written as $d/dx(Iu)$ as a result of the product rule.

$$\frac{d}{dx}\left(\frac{1}{x}u\right) = -x^2$$

Integrate both sides with respect to x.

$$\frac{1}{x}u = -\frac{1}{3}x^3 + C$$

Multiply both sides to solve for u.

$$u(x) = -\frac{1}{3}x^4 + Cx$$

Change back now to the original variable y.

$$\frac{1}{y} = -\frac{1}{3}x^4 + Cx$$

Invert both sides to solve for y.

$$y(x) = \frac{1}{-\frac{1}{3}x^4 + Cx}$$

Multiply the numerator and denominator by 3 and factor the x in the denominator.

$$y(x) = \frac{3}{x(3C - x^3)}.$$

Use a new arbitrary constant for $3C$ to get the final result.

$$y(x) = \frac{3}{x(A - x^3)}$$