Problem 1.35

An Abel equation has the general form $y' = a(x) + b(x)y + c(x)y^2 + d(x)y^3$. Solve the particular equation $y' = dy^3 + ax^{-3/2}$, where d and a are constants.

Solution

Multiply both sides of the equation by $x^{3/2}$.

$$x^{3/2}\frac{dy}{dx} = dx^{3/2}y^3 + a \tag{1}$$

Make the following scale transformations.

$$x \to bx$$
$$y \to b^p y$$

If there is a value of p that leaves the ODE unchanged, then it is said to be scale invariant, and progress can be made in solving it.

$$(bx)^{3/2}\frac{d(b^p y)}{d(bx)} = d(bx)^{3/2}(b^p y)^3 + a$$

Pull the constants out of the derivative and separate the b terms from x and y.

$$b^{3/2}x^{3/2}\frac{b^p}{b}\frac{dy}{dx} = db^{3/2}x^{3/2}b^{3p}y^3 + a$$

Combine the b terms on both sides.

$$b^{p+1/2}x^{3/2}\frac{dy}{dx} = b^{3(p+1/2)}dx^{3/2}y^3 + a$$

Notice that if $p = -1/2$, then the b terms go away and we're back to equation (1). Thus, the ODE is scale invariant under the transformation, $x \to bx$, $y \to b^{-1/2}y$. This means we can make the substitution,

$$y(x) = x^{-1/2}u(x),$$

to make the ODE equidimensional. Take the derivative to find out what y' is in terms of u.

$$\frac{dy}{dx} = -\frac{1}{2}x^{-3/2}u + x^{-1/2}\frac{du}{dx}$$

Substitute these expressions into equation (1).

$$x^{3/2}\left(-\frac{1}{2}x^{-3/2}u + x^{-1/2}\frac{du}{dx}\right) = dx^{3/2}(x^{-1/2}u)^3 + a$$

Expand both sides.

$$x\frac{du}{dx} - \frac{1}{2}u = du^3 + a \tag{2}$$

This is a first-order equation that can be solved with separation of variables.

$$x\frac{du}{dx} = du^3 + \frac{1}{2}u + a.$$

Separate variables.

$$\frac{du}{du^3 + \frac{1}{2}u + a} = \frac{dx}{x}$$

Integrate both sides.

$$\int^u \frac{ds}{ds^3 + \frac{1}{2}s + a} = \ln|x| + C$$

Because of the term with $x^{3/2}$ in the ODE we started with, x is implied to be greater than zero, so the absolute value sign can be removed. Now that the integration is done, change back to the original variable y.

$$\int^{\sqrt{x}y} \frac{ds}{ds^3 + \frac{1}{2}s + a} = \ln x + C$$

This is only an implicit solution for y but a solution nonetheless.

From equation (2), one can make the exponential substitution, $x = e^t$, to make the ODE autonomous.

$$\frac{du}{dt} - \frac{1}{2}u = du^3 + a$$

Perhaps there is an explicit solution for u, but I can't see it.