Problem 1.39

Show that any equation of the form (1.8.9) can be transformed to Sturm-Liouville form (1.8.10).

Solution

The goal here is to show that the second-order eigenvalue problem,

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) + d(x)Ey(x) = 0, \tag{1}$$

can be transformed to Sturm-Liouville form,

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + Er(x)]y = 0,$$

by choosing $p(x)$, $q(x)$, and $r(x)$ appropriately. Start by dividing both sides of equation (1) by $a(x)$, which we assume is not zero.

$$y''(x) + \frac{b(x)}{a(x)}y'(x) + \frac{c(x)}{a(x)}y(x) + \frac{d(x)}{a(x)}Ey(x) = 0$$

Multiply both sides by the integrating factor I.

$$I = e^{\int^x \frac{b(s)}{a(s)}\,ds}$$

We get the following ODE.

$$e^{\int^x \frac{b(s)}{a(s)}\,ds}y''(x) + \frac{b(x)}{a(x)}e^{\int^x \frac{b(s)}{a(s)}\,ds}y'(x) + \frac{c(x)}{a(x)}e^{\int^x \frac{b(s)}{a(s)}\,ds}y(x) + \frac{d(x)}{a(x)}e^{\int^x \frac{b(s)}{a(s)}\,ds}Ey(x) = 0$$

The first two terms on the left can now be written as $d/dx(Iy')$ as a result of the product rule. Factor $y(x)$ from the last two terms.

$$\frac{d}{dx}\left[e^{\int^x \frac{b(s)}{a(s)}\,ds}y'(x)\right] + \left[\frac{c(x)}{a(x)}e^{\int^x \frac{b(s)}{a(s)}\,ds} + \frac{d(x)}{a(x)}e^{\int^x \frac{b(s)}{a(s)}\,ds}E\right]y(x) = 0$$

Therefore, by choosing

$$\begin{aligned}
p(x) &= e^{\int^x \frac{b(s)}{a(s)}\,ds} \\
q(x) &= \frac{c(x)}{a(x)}e^{\int^x \frac{b(s)}{a(s)}\,ds} \\
r(x) &= \frac{d(x)}{a(x)}e^{\int^x \frac{b(s)}{a(s)}\,ds},
\end{aligned}$$

equation (1) can be transformed to Sturm-Liouville form.