Problem 1.40

Verify (1.9.1).

Solution

(1.9.1) says that if y_1 and y_2 are two linearly independent solutions to the complex ODE, $y'' + p_1(z)y' + p_0(z)y = 0$, then

$$p_0(z) = \frac{y_1''y_2' - y_2''y_1'}{y_1'y_2 - y_1y_2'}, \quad p_1(z) = \frac{y_2''y_1 - y_1''y_2}{y_1'y_2 - y_1y_2'}.$$

If y_1 and y_2 are both solutions, then they must both satisfy the ODE.

$$y_1'' + p_1(z)y_1' + p_0(z)y_1 = 0 \tag{1}$$
$$y_2'' + p_1(z)y_2' + p_0(z)y_2 = 0 \tag{2}$$

We have two equations, so we can solve for the two unknowns, $p_0(z)$ and $p_1(z)$, with substitution. Solve equation (1) for $p_0(z)$.

$$p_0(z) = -\frac{y_1'' + p_1(z)y_1'}{y_1} \tag{3}$$

Now substitute this expression for $p_0(z)$ into equation (2).

$$y_2'' + p_1(z)y_2' - \frac{y_2}{y_1}[y_1'' + p_1(z)y_1'] = 0$$

Multiply both sides by y_1.

$$y_1y_2'' + p_1(z)y_1y_2' - y_1''y_2 - p_1(z)y_1'y_2 = 0$$

Move the terms without $p_1(z)$ to the right side and factor $p_1(z)$ on the left side.

$$p_1(z)(y_1y_2' - y_1'y_2) = y_1''y_2 - y_1y_2''$$

Divide both sides by $y_1y_2' - y_1'y_2$ to solve for $p_1(z)$.

$$p_1(z) = \frac{y_1''y_2 - y_1y_2''}{y_1y_2' - y_1'y_2}$$

Multiply the numerator and denominator by -1 to obtain the desired result for $p_1(z)$.

$$\boxed{p_1(z) = \frac{y_1y_2'' - y_1''y_2}{y_1'y_2 - y_1y_2'}}$$

To solve for $p_0(z)$, use equation (3).

$$p_0(z) = -\frac{y_1''}{y_1} - \frac{y_1'}{y_1}p_1(z)$$

Plug in the expression we found for $p_1(z)$.

$$p_0(z) = -\frac{y_1''}{y_1} - \frac{y_1'}{y_1} \cdot \frac{y_1y_2'' - y_1''y_2}{y_1'y_2 - y_1y_2'}$$

In order to obtain a common denominator to combine the fractions, multiply the numerator and denominator of the first fraction by $y_1'y_2 - y_1y_2'$.

$$p_0(z) = -\frac{y_1''(y_1'y_2 - y_1y_2')}{y_1(y_1'y_2 - y_1y_2')} - \frac{y_1y_1'y_2'' - y_1'y_1''y_2}{y_1(y_1'y_2 - y_1y_2')}$$

Expand the numerator of the first fraction and distribute the minus signs.

$$p_0(z) = \frac{-y_1'y_1''y_2 + y_1y_1''y_2'}{y_1(y_1'y_2 - y_1y_2')} + \frac{y_1'y_1''y_2 - y_1y_1'y_2''}{y_1(y_1'y_2 - y_1y_2')}$$

Combine the fractions.

$$p_0(z) = \frac{\cancel{-y_1'y_1''y_2} + y_1y_1''y_2' + \cancel{y_1'y_1''y_2} - y_1y_1'y_2''}{y_1(y_1'y_2 - y_1y_2')}$$

Cancel y_1 from the numerator and the denominator.

$$p_0(z) = \frac{\cancel{y_1}y_1''y_2' - \cancel{y_1}y_1'y_2''}{\cancel{y_1}(y_1'y_2 - y_1y_2')}$$

Therefore,

$$\boxed{p_0(z) = \frac{y_1''y_2' - y_1'y_2''}{y_1'y_2 - y_1y_2'}.}$$