Problem 14

In each of Problems 11 through 14, draw a direction field for the given differential equation. Based on the direction field, determine the behavior of \( y \) as \( t \to \infty \). If this behavior depends on the initial value of \( y \) at \( t = 0 \), describe this dependency. Note that in these problems the equations are not of the form \( y' = ay + b \), and the behavior of their solutions is somewhat more complicated than for the equations in the text.

\[
y' = y(y - 2)^2
\]

Solution

The direction field is a two-dimensional vector field that shows what the direction of the solution is at every point in a region. Every solution to the differential equation is a curve drawn such that the direction field vectors are tangent to it at every point.

\[
\langle dt, dy \rangle = \left\langle 1, \frac{dy}{dt} \right\rangle dt = \left\langle 1, y(y - 2)^2 \right\rangle dt
\]

Figure 1: In blue are the direction field vectors and in red are possible solutions to the differential equation, depending what the initial condition is.
The equilibrium solution is found by setting \( y' = 0 \) in the differential equation and solving the resulting equation for \( y \).

\[
0 = y(y - 2)^2
\]

\[
y = 0 \quad \text{or} \quad y = 2
\]

Notice from Figure 1 that the behavior of \( y \) as \( t \to \infty \) depends on what the initial condition is. If \( y(t = 0) > 2 \), then the solution diverges to \( \infty \) as \( t \to \infty \). If \( 0 < y(t = 0) < 2 \), then the solution converges to \( y = 2 \) as \( t \to \infty \). If \( y(t = 0) < 0 \), then the solution diverges to \( -\infty \) as \( t \to \infty \).