Problem 25

For small, slowly falling objects, the assumption made in the text that the drag force is proportional to the velocity is a good one. For larger, more rapidly falling objects, it is more accurate to assume that the drag force is proportional to the square of the velocity.[2]

(a) Write a differential equation for the velocity of a falling object of mass m if the magnitude of the drag force is proportional to the square of the velocity and its direction is opposite to that of the velocity.

(b) Determine the limiting velocity after a long time.

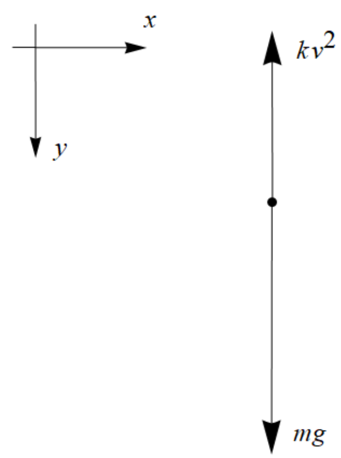
(c) If $m = 10$ kg, find the drag coefficient so that the limiting velocity is 49 m/s.

(d) Using the data in part (c), draw a direction field and compare it with Figure 1.1.3.

Solution

Part (a)

Let k be the proportionality constant so that the drag force is $F_d = kv^2$. There are two forces acting on a falling object, the gravitational force mg and the resistive drag force kv^2, as illustrated in the free-body diagram below.



Newton's second law states that the sum of the forces is equal to mass times acceleration.

$$\sum \mathbf{F} = m\mathbf{a}$$

This vector equation represents the following two scalar equations in the chosen coordinate system.

$$\sum F_x = ma_x$$
$$\sum F_y = ma_y$$

[2]See Lyle N. Long and Howard Weiss, "The Velocity Dependence of Aerodynamic Drag: A Primer for Mathematicians," *American Mathematical Monthly 106* (1999), 2, pp. 127–135.

Apply Newton's law to the falling object.

$$0 = 0$$
$$mg - kv^2 = ma$$

Acceleration is the time rate of change of velocity.

$$mg - kv^2 = m\frac{dv}{dt}$$

Therefore,

$$\frac{dv}{dt} = g - \frac{k}{m}v^2.$$

Part (b)

After a long time the drag force matches the gravitational force in magnitude, and the object's velocity remains the same. That is, $dv/dt \to 0$ as $t \to \infty$.

$$0 = g - \frac{k}{m}v^2$$
$$\frac{k}{m}v^2 = g$$
$$v^2 = \frac{mg}{k}$$

Therefore, the equilibrium solution for the velocity is

$$v = \sqrt{\frac{mg}{k}}.$$

Part (c)

Solve the previous equation for the drag coefficient k.

$$k = \frac{mg}{v^2}$$

If $m = 10$ kg and $v = 49$ m/s, the drag coefficient is

$$k = \frac{10g}{2401}\ \frac{\text{kg}}{\text{m}} \approx 0.0408\ \frac{\text{kg}}{\text{m}}.$$

Part (d)

With $m = 10$ kg and $k = 10g/2401$ kg/m, the differential equation for the velocity becomes

$$\frac{dv}{dt} = g - \frac{g}{2401}v^2. \tag{1}$$

The direction field is a two-dimensional vector field that shows what the direction of the solution is at every point in a region. Every solution to the differential equation is a curve drawn such that the direction field vectors are tangent to it at every point.

$$\langle dt, dv\rangle = \left\langle 1, \frac{dv}{dt}\right\rangle dt = \left\langle 1, g - \frac{g}{2401}v^2\right\rangle dt$$

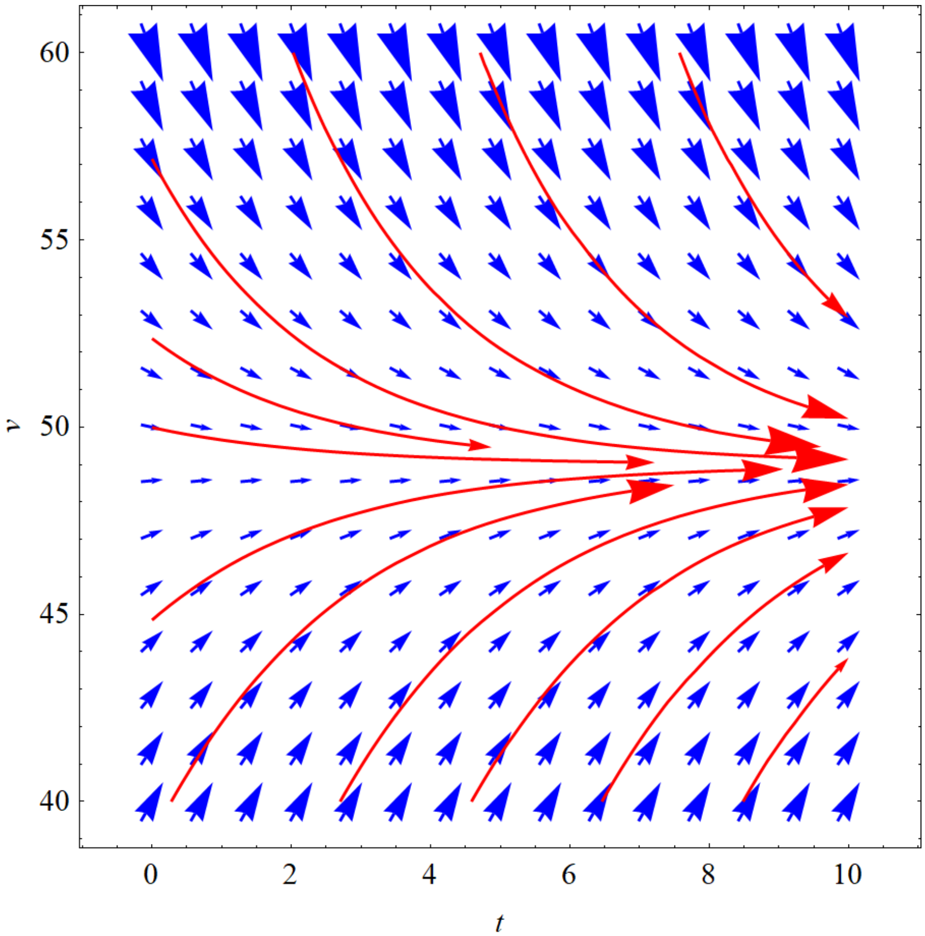


Figure 1: In blue are the direction field vectors and in red are possible solutions to equation (1), depending what the initial condition is. The nonequilibrium solutions appear to converge to $v = 49$ as $t \to \infty$.

Below in Figure 2 is Figure 1.1.3, the direction field of

$$\frac{dv}{dt} = g - \frac{v}{5}. \tag{2}$$

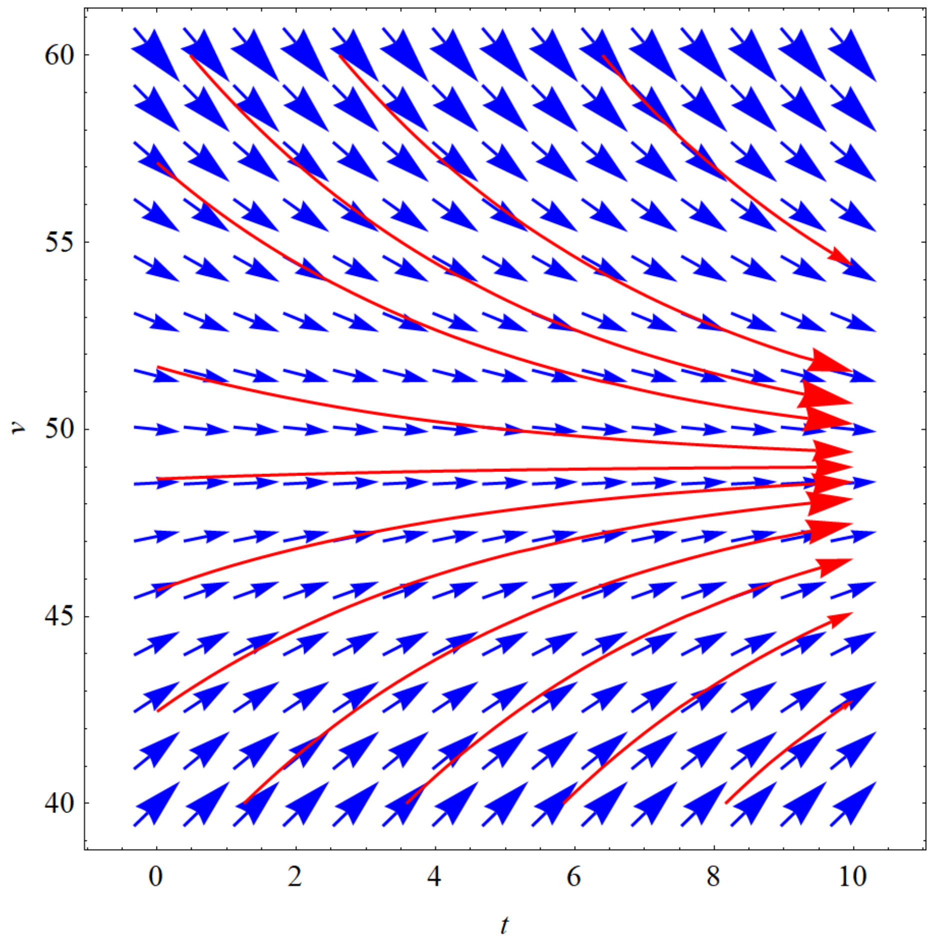


Figure 2: In blue are the direction field vectors and in red are possible solutions to equation (2), depending what the initial condition is. The nonequilibrium solutions appear to converge to $v = 49$ as $t \to \infty$.

The solution curves of equation (2) converge more slowly to $v = 49$ than those of equation (1).