Problem 25

For small, slowly falling objects, the assumption made in the text that the drag force is proportional to the velocity is a good one. For larger, more rapidly falling objects, it is more accurate to assume that the drag force is proportional to the square of the velocity.²

(a) Write a differential equation for the velocity of a falling object of mass \( m \) if the magnitude of the drag force is proportional to the square of the velocity and its direction is opposite to that of the velocity.

(b) Determine the limiting velocity after a long time.

(c) If \( m = 10 \text{ kg} \), find the drag coefficient so that the limiting velocity is 49 m/s.

(d) Using the data in part (c), draw a direction field and compare it with Figure 1.1.3.

Solution

Part (a)

Let \( k \) be the proportionality constant so that the drag force is \( F_d = kv^2 \). There are two forces acting on a falling object, the gravitational force \( mg \) and the resistive drag force \( kv^2 \), as illustrated in the free-body diagram below.

\[
\sum \mathbf{F} = ma
\]

Newton’s second law states that the sum of the forces is equal to mass times acceleration.

This vector equation represents the following two scalar equations in the chosen coordinate system.

\[
\sum F_x = ma_x
\]
\[
\sum F_y = ma_y
\]


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Apply Newton’s law to the falling object.

\[ 0 = 0 \]
\[ mg - kv^2 = ma \]

Acceleration is the time rate of change of velocity.

\[ mg - kv^2 = m \frac{dv}{dt} \]

Therefore,

\[ \frac{dv}{dt} = g - \frac{k}{m}v^2. \]

**Part (b)**

After a long time the drag force matches the gravitational force in magnitude, and the object’s velocity remains the same. That is, \( \frac{dv}{dt} \to 0 \) as \( t \to \infty \).

\[ 0 = g - \frac{k}{m}v^2 \]
\[ \frac{k}{m}v^2 = g \]
\[ v^2 = \frac{mg}{k} \]

Therefore, the equilibrium solution for the velocity is

\[ v = \sqrt{\frac{mg}{k}}. \]

**Part (c)**

Solve the previous equation for the drag coefficient \( k \).

\[ k = \frac{mg}{v^2} \]

If \( m = 10 \) kg and \( v = 49 \) m/s, the drag coefficient is

\[ k = \frac{10g}{2401} \text{ kg/m} \approx 0.0408 \text{ kg/m}. \]

**Part (d)**

With \( m = 10 \) kg and \( k = 10g/2401 \) kg/m, the differential equation for the velocity becomes

\[ \frac{dv}{dt} = g - \frac{g}{2401}v^2. \]

The direction field is a two-dimensional vector field that shows what the direction of the solution is at every point in a region. Every solution to the differential equation is a curve drawn such that the direction field vectors are tangent to it at every point.

\[ \langle dt, dv \rangle = \left\langle 1, \frac{dv}{dt} \right\rangle dt = \left\langle 1, g - \frac{g}{2401}v^2 \right\rangle dt \]

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Figure 1: In blue are the direction field vectors and in red are possible solutions to equation (1), depending what the initial condition is. The nonequilibrium solutions appear to converge to $v = 49$ as $t \to \infty$. 

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Below in Figure 2 is Figure 1.1.3, the direction field of

\[
\frac{dv}{dt} = g - \frac{v}{5}.
\]

(2)

Figure 2: In blue are the direction field vectors and in red are possible solutions to equation (2), depending what the initial condition is. The nonequilibrium solutions appear to converge to \( v = 49 \) as \( t \to \infty \).

The solution curves of equation (2) converge more slowly to \( v = 49 \) than those of equation (1).