Problem 3

Consider the differential equation

$$dy/dt = -ay + b,$$

where a and b are positive numbers.

(a) Find the general solution of the differential equation.

(b) Sketch the solution for several different initial conditions.

(c) Describe how the solutions change under each of the following conditions:

i. a increases.
ii. b increases.
iii. Both a and b increase, but the ratio b/a remains the same.

Solution

Part (a)

$$\begin{aligned} y' &= -ay + b \\ &= -a\left(y - \frac{b}{a}\right) \end{aligned}$$

Divide both sides by $y - b/a$.

$$\frac{y'}{y - \frac{b}{a}} = -a$$

The left side can be written as $d/dt(\ln|y - b/a|)$ by the chain rule. The absolute value sign is included because the argument of the logarithm cannot be negative.

$$\frac{d}{dt}\ln\left|y - \frac{b}{a}\right| = -a$$

Integrate both sides with respect to t.

$$\ln\left|y - \frac{b}{a}\right| = -at + C$$

Exponentiate both sides.

$$\begin{aligned} \left|y - \frac{b}{a}\right| &= e^{-at+C} \\ &= e^C e^{-at} \end{aligned}$$

Introduce $\pm$ on the right side in order to remove the absolute value sign.

$$y - \frac{b}{a} = \pm e^C e^{-at}$$

Let $A = \pm e^C$ and add b/a to both sides to solve for y.

$$y(t) = \frac{b}{a} + Ae^{-at}$$

Part (b)

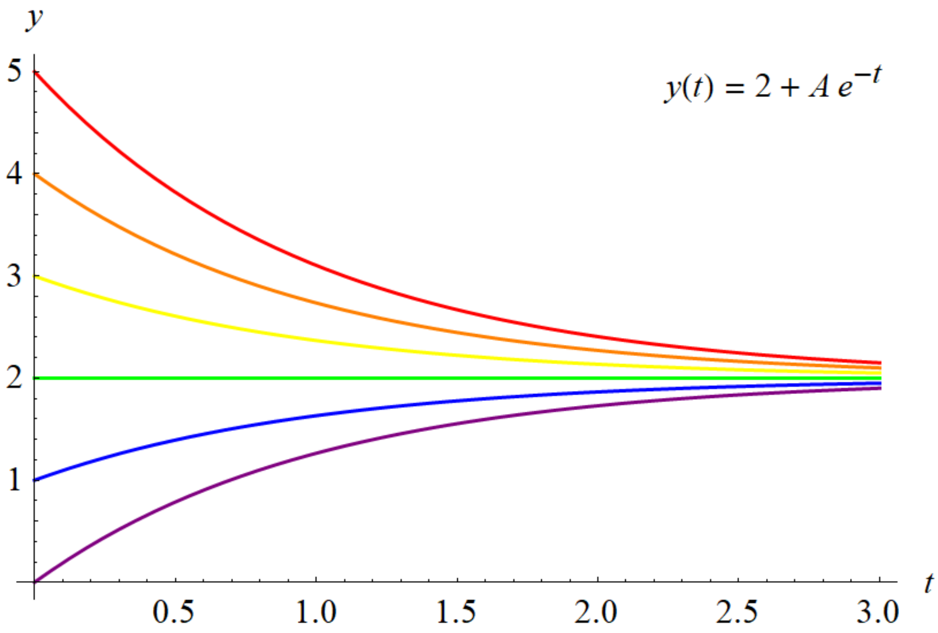

Figure 1: In this figure $y(t)$ is plotted versus t for $a = 1$, $b = 2$, and various values of A. The curves in red, orange, yellow, green, blue, and purple correspond to $A = 3$, $A = 2$, $A = 1$, $A = 0$, $A = -1$, and $A = -2$, respectively. The equilibrium solution is at $y = b/a = 2$.

Part (c)

i. If a increases, then the solutions will decay even faster than before. Also, the equilibrium solution will become smaller.

ii. If b increases, then the equilibrium solution will become bigger.

iii. If a and b increase but the ratio b/a remains the same, then the equilibrium solution will be the same but the solutions will decay faster than before.