Problem 8

Consider a population \( p \) of field mice that grows at a rate proportional to the current population, so that \( \frac{dp}{dt} = rp \).

(a) Find the rate constant \( r \) if the population doubles in 30 days.

(b) Find \( r \) if the population doubles in \( N \) days.

Solution

Note that \( t \) is measured in days in this problem.

\[ p' = rp \]

Divide both sides by \( p \).

\[ \frac{p'}{p} = r \]

The left side can be written as \( \frac{d}{dt}(\ln |p|) \) by the chain rule. The absolute value sign is included because the argument of the logarithm cannot be negative.

\[ \frac{d}{dt} \ln |p| = r \]

Integrate both sides with respect to \( t \).

\[ \ln |p| = rt + C \]

Exponentiate both sides.

\[ |p| = e^{rt+C} = e^C e^{rt} \]

Introduce \( \pm \) on the right side to remove the absolute value sign.

\[ p(t) = \pm e^{C} e^{rt} \]

Let \( A = \pm e^{C} \).

\[ p(t) = Ae^{rt} \]

Apply the initial condition \( p(0) = p_0 \), where \( p_0 \) is the initial mouse population, to determine \( A \).

\[ p(0) = A = p_0 \]

Therefore, the general solution to the ODE is

\[ p(t) = p_0 e^{rt}. \]
Part (a)

In order to find the rate constant $r$ if the population doubles in 30 days, set $p = 2p_0$ and $t = 30$ days and solve the resulting equation for $r$.

\[
2p_0 = p_0 e^{30r} \\
2 = e^{30r} \\
\ln 2 = \ln e^{30r} \\
\ln 2 = 30r \ln e \\
\ln 2 = 30r
\]

Therefore,

\[r = \frac{\ln 2}{30} \approx 0.0231 \text{ day}^{-1}.
\]

$r$ is in units of $1/\text{day}$ because the exponent of $e$ must be dimensionless.

Part (b)

In order to find the rate constant $r$ if the population doubles in $N$ days, set $p = 2p_0$ and $t = N$ and solve the resulting equation for $r$.

\[
2p_0 = p_0 e^{rN} \\
2 = e^{rN} \\
\ln 2 = \ln e^{rN} \\
\ln 2 = rN \ln e \\
\ln 2 = rN
\]

Therefore,

\[r = \frac{\ln 2}{N}.
\]