Problem 10

Modify Example 2 so that the falling object experiences no air resistance.

(a) Write down the modified initial value problem.

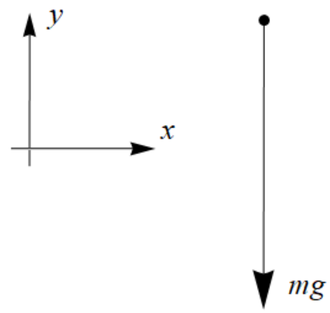
(b) Determine how long it takes the object to reach the ground.

(c) Determine its velocity at the time of impact.

Solution

Part (a)

Neglecting air resistance, there is one force acting on a falling object, the gravitational force mg, as illustrated in the free-body diagram below.



Newton's second law states that the force is equal to mass times acceleration.

$$\mathbf{F} = m\mathbf{a}$$

This vector equation represents the following two scalar equations in the chosen coordinate system.

$$\begin{aligned} F_x &= ma_x \\ F_y &= ma_y \end{aligned}$$

Apply Newton's law to the falling object.

$$\begin{aligned} 0 &= 0 \\ -mg &= ma \end{aligned}$$

Acceleration is the rate of change of velocity with respect to time.

$$-mg = m\frac{dv}{dt}$$

Divide both sides by m to obtain the governing equation for the velocity.

$$\frac{dv}{dt} = -g$$

Assuming the object starts from rest, the initial condition associated with this ODE is $v(0) = 0$. Therefore, the modified initial value problem is

$$\frac{dv}{dt} = -g, \quad v(0) = 0.$$

Part (b)

Integrate both sides of the ODE with respect to t.

$$v(t) = -gt + C_1$$

Apply the initial condition to solve for C_1.

$$v(0) = C_1 = 0$$

So the velocity is

$$v(t) = -gt.$$

Velocity is the rate of change of position with respect to time.

$$\frac{dx}{dt} = v(t)$$

To solve for x, integrate both sides with respect to t.

$$\begin{aligned} x(t) &= \int v(t)\,dt + C_2 \\ &= \int (-gt)\,dt + C_2 \\ &= -\frac{1}{2}gt^2 + C_2 \end{aligned}$$

According to Example 2, the object is initially 300 meters in the air, so the initial condition is $x(0) = 300$ m. Apply the initial condition now to determine C_2.

$$x(0) = C_2 = 300$$

Therefore,

$$x(t) = -\frac{1}{2}gt^2 + 300.$$

In order to find how long it takes the object to reach the ground, set $x = 0$ and solve the resulting equation for t.

$$\begin{aligned} 0 &= -\frac{1}{2}gt^2 + 300 \\ \frac{1}{2}gt^2 &= 300 \\ t^2 &= \frac{600}{g} \\ t &= \sqrt{\frac{600}{g}} \approx 7.82 \text{ s} \end{aligned}$$

Part (c)

To find the velocity of the object at impact, evaluate $v(\sqrt{600/g})$.

$$v\left(\sqrt{600}g\right) = -g\sqrt{\frac{600}{g}} \approx -76.68\ \frac{\text{m}}{\text{s}}$$

Therefore, the object is travelling 76.68 m/s downward as it hits the ground.