Problem 13

The **half-life** of a radioactive material is the time required for an amount of this material to decay to one-half its original value. Show that for any radioactive material that decays according to the equation \( Q' = -rQ \), the half-life \( \tau \) and the decay rate \( r \) satisfy the equation \( r\tau = \ln 2 \).

**Solution**

\[
Q' = -rQ
\]

Divide both sides by \( Q \).

\[
\frac{Q'}{Q} = -r
\]

The left side can be written as \( \frac{d}{dt}(\ln Q) \) by the chain rule.

\[
\frac{d}{dt} \ln Q = -r
\]

Integrate both sides with respect to \( t \).

\[
\ln Q = -rt + C
\]

Exponentiate both sides.

\[
Q(t) = e^{\ln Q} = e^{-rt + C} = e^C e^{-rt}
\]

Let \( A = e^C \).

\[
Q(t) = Ae^{-rt}
\]

Suppose there is a mass of \( Q_0 \) initially so that the initial condition is \( Q(0) = Q_0 \). Then the constant \( A \) can be determined.

\[
Q(0) = A = Q_0
\]

As a result,

\[
Q(t) = Q_0 e^{-rt}.
\]

After one half-life has passed, only half of the initial mass remains by definition.

\[
Q(\tau) = \frac{Q_0}{2} = Q_0 e^{-r\tau}
\]

\[
e^{-r\tau} = \frac{1}{2}
\]

\[
\ln e^{-r\tau} = \ln \frac{1}{2}
\]

\[
-r\tau \ln e = -\ln 2
\]

\[
-r\tau = -\ln 2
\]

Therefore,

\[
r\tau = \ln 2.
\]