Problem 13

The **half-life** of a radioactive material is the time required for an amount of this material to decay to one-half its original value. Show that for any radioactive material that decays according to the equation $Q' = -rQ$, the half-life τ and the decay rate r satisfy the equation $r\tau = \ln 2$.

Solution

$$Q' = -rQ$$

Divide both sides by Q.

$$\frac{Q'}{Q} = -r$$

The left side can be written as $d/dt(\ln Q)$ by the chain rule.

$$\frac{d}{dt}\ln Q = -r$$

Integrate both sides with respect to t.

$$\ln Q = -rt + C$$

Exponentiate both sides.

$$\begin{aligned} Q(t) &= e^{-rt+C} \\ &= e^C e^{-rt} \end{aligned}$$

Let $A = e^C$.

$$Q(t) = Ae^{-rt}$$

Suppose there is a mass of Q_0 initially so that the initial condition is $Q(0) = Q_0$. Then the constant A can be determined.

$$Q(0) = A = Q_0$$

As a result,

$$Q(t) = Q_0 e^{-rt}.$$

After one half-life has passed, only half of the initial mass remains by definition.

$$\begin{aligned} Q(\tau) = \frac{Q_0}{2} &= Q_0 e^{-r\tau} \\ e^{-r\tau} &= \frac{1}{2} \\ \ln e^{-r\tau} &= \ln \frac{1}{2} \\ -r\tau \ln e &= -\ln 2 \\ -r\tau &= -\ln 2 \end{aligned}$$

Therefore,

$$r\tau = \ln 2.$$