Problem 14

Radium-226 has a half-life of 1620 years. Find the time period during which a given amount of this material is reduced by one-quarter.

Solution

Assuming that radium-226 decays at a rate proportional to the amount present at any particular time, it obeys the relationship,

\[ Q' = -rQ, \]

where \( r > 0 \) is the decay rate. Divide both sides by \( Q \).

\[ \frac{Q'}{Q} = -r \]

The left side can be written as \( d/dt(\ln Q) \) by the chain rule.

\[ \frac{d}{dt} \ln Q = -r \]

Integrate both sides with respect to \( t \).

\[ \ln Q = -rt + C \]

Exponentiate both sides.

\[ Q(t) = e^{-rt+C} = e^C e^{-rt} \]

Let \( A = e^C \).

\[ Q(t) = Ae^{-rt} \]

Suppose there is a mass of \( Q_0 \) initially so that the initial condition is \( Q(0) = Q_0 \). Then the constant \( A \) can be determined.

\[ Q(0) = A = Q_0 \]

As a result,

\[ Q(t) = Q_0e^{-rt}. \]

By definition the half-life \( \tau \) is defined to be the amount of time it takes for half the amount of some mass to decay. \( r \) can be found by setting \( Q = Q_0/2 \) and \( t = 1620 \) and solving the resulting equation for it.

\[
\begin{align*}
\frac{Q_0}{2} &= Q_0e^{-1620r} \\
\frac{1}{2} &= e^{-1620r} \\
\ln \frac{1}{2} &= \ln e^{-1620r} \\
-\ln 2 &= -1620r \ln e \\
\ln 2 &= 1620r \\
r &= \frac{\ln 2}{1620}
\end{align*}
\]
Consequently, the formula for $Q$ becomes

$$Q(t) = Q_0 \exp \left( -\frac{\ln 2}{1620} t \right).$$

Now that $r$ is known, the mass at any time can be calculated. In order to find the time period during which a given amount of this material is reduced by one-quarter, set $Q = 0.75Q_0$ and solve the resulting equation for $t$.

\[
0.75Q_0 = Q_0 \exp \left( -\frac{\ln 2}{1620} t \right) \\
0.75 = \exp \left( -\frac{\ln 2}{1620} t \right) \\
\ln 0.75 = -\frac{\ln 2}{1620} t \\
t = -1620 \frac{\ln 0.75}{\ln 2} \approx 672.4 \text{ years}
\]