Problem 14

Radium-226 has a half-life of 1620 years. Find the time period during which a given amount of this material is reduced by one-quarter.

Solution

Assuming that radium-226 decays at a rate proportional to the amount present at any particular time, it obeys the relationship,

$$Q' = -rQ,$$

where $r > 0$ is the decay rate. Divide both sides by Q.

$$\frac{Q'}{Q} = -r$$

The left side can be written as $d/dt(\ln Q)$ by the chain rule.

$$\frac{d}{dt} \ln Q = -r$$

Integrate both sides with respect to t.

$$\ln Q = -rt + C$$

Exponentiate both sides.

$$\begin{aligned} Q(t) &= e^{-rt+C} \\ &= e^C e^{-rt} \end{aligned}$$

Let $A = e^C$.

$$Q(t) = Ae^{-rt}$$

Suppose there is a mass of Q_0 initially so that the initial condition is $Q(0) = Q_0$. Then the constant A can be determined.

$$Q(0) = A = Q_0$$

As a result,

$$Q(t) = Q_0 e^{-rt}.$$

By definition the half-life τ is defined to be the amount of time it takes for half the amount of some mass to decay. r can be found by setting $Q = Q_0/2$ and $t = 1620$ and solving the resulting equation for it.

$$\begin{aligned} \frac{Q_0}{2} &= Q_0 e^{-1620r} \\ \frac{1}{2} &= e^{-1620r} \\ \ln \frac{1}{2} &= \ln e^{-1620r} \\ -\ln 2 &= -1620r \ln e \\ \ln 2 &= 1620r \\ r &= \frac{\ln 2}{1620} \end{aligned}$$

Consequently, the formula for Q becomes

$$Q(t) = Q_0 \exp\left(-\frac{\ln 2}{1620}t\right).$$

Now that r is known, the mass at any time can be calculated. In order to find the time period during which a given amount of this material is reduced by one-quarter, set $Q = 0.75Q_0$ and solve the resulting equation for t.

$$\begin{aligned}
0.75Q_0 &= Q_0 \exp\left(-\frac{\ln 2}{1620}t\right) \\
0.75 &= \exp\left(-\frac{\ln 2}{1620}t\right) \\
\ln 0.75 &= -\frac{\ln 2}{1620}t \\
t &= -1620\frac{\ln 0.75}{\ln 2} \approx 672.4 \text{ years}
\end{aligned}$$