Problem 15

According to Newton's law of cooling (see Problem 23 of Section 1.1), the temperature $u(t)$ of an object satisfies the differential equation

$$\frac{du}{dt} = -k(u - T),$$

where T is the constant ambient temperature and k is a positive constant. Suppose that the initial temperature of the object is $u(0) = u_0$.

(a) Find the temperature of the object at any time.

(b) Let τ be the time at which the initial temperature difference $u_0 - T$ has been reduced by half. Find the relation between k and τ.

Solution

Part (a)

$$u' = -k(u - T)$$

Divide both sides by $u - T$.

$$\frac{u'}{u - T} = -k$$

The left side can be written as $d/dt(\ln |u - T|)$ by the chain rule. The absolute value sign is included because the argument of the logarithm cannot be negative.

$$\frac{d}{dt} \ln |u - T| = -k$$

Integrate both sides with respect to t.

$$\ln |u - T| = -kt + C$$

Exponentiate both sides.

$$\begin{aligned} |u - T| &= e^{-kt+C} \\ &= e^C e^{-kt} \end{aligned}$$

Introduce $\pm$ on the right side to remove the absolute value sign.

$$u(t) - T = \pm e^C e^{-kt}$$

Let $A = \pm e^C$ and add T to both sides to solve for $u(t)$.

$$u(t) = T + Ae^{-kt}$$

Apply the initial condition here to determine A.

$$u(0) = T + A = u_0 \quad \rightarrow \quad A = u_0 - T$$

Therefore,

$$u(t) = T + (u_0 - T)e^{-kt}.$$

This solution can also be expressed as

$$\frac{u(t)-T}{u_0-T}=e^{-kt}.$$

Part (b)

If τ is the time at which the initial temperature difference u_0-T has been reduced by half, then

$$\begin{aligned}\frac{u(\tau)-T}{u_0-T}&=e^{-k\tau}\\ \frac{\frac{1}{2}(u_0-T)}{u_0-T}&=e^{-k\tau}\\ \frac{1}{2}&=e^{-k\tau}\\ \ln\frac{1}{2}&=\ln e^{-k\tau}\\ -\ln 2&=-k\tau\ln e.\end{aligned}$$

Therefore,

$$k\tau=\ln 2.$$