Problem 16

Suppose that a building loses heat in accordance with Newton's law of cooling (see Problem 15) and that the rate constant k has the value 0.15 h^{-1}. Assume that the interior temperature is 70°F when the heating system fails. If the external temperature is 10°F, how long will it take for the interior temperature to fall to 32°F?

Solution

The initial value problem to solve here is

$$\frac{du}{dt} = -0.15(u - 10), \quad u(0) = 70.$$

Divide both sides of the ODE by $u - 10$.

$$\frac{u'}{u - 10} = -0.15$$

The left side can be written as $d/dt[\ln(u - 10)]$ by the chain rule.

$$\frac{d}{dt}\ln(u - 10) = -0.15$$

Integrate both sides with respect to time.

$$\ln(u - 10) = -0.15t + C$$

Exponentiate both sides.

$$\begin{aligned} u(t) - 10 &= e^{-0.15t+C} \\ &= e^{C}e^{-0.15t} \end{aligned}$$

Let $A = e^C$ and add 10 to both sides to solve for $u(t)$.

$$u(t) = 10 + Ae^{-0.15t}$$

Apply the initial condition here to determine A.

$$u(0) = 10 + A = 70 \quad \rightarrow \quad A = 60$$

As a result, the temperature at any time is

$$u(t) = 10 + 60e^{-0.15t}$$

In order to find how long it takes for the interior temperature to fall to 32°F, set $u = 32$ and solve the resulting equation for t.

$$\begin{aligned}
32 &= 10 + 60e^{-0.15t} \\
22 &= 60e^{-0.15t} \\
\frac{11}{30} &= e^{-0.15t} \\
\ln e^{-0.15t} &= \ln \frac{11}{30} \\
-0.15t \ln e &= -\ln \frac{30}{11} \\
\frac{3}{20} t &= \ln \frac{30}{11} \\
t = \frac{20}{3} \ln \frac{30}{11} &\approx 6.69 \text{ hours}
\end{aligned}$$