Problem 19

In each of Problems 19 and 20, determine the values of r for which the given differential equation has solutions of the form $y = t^r$ for $t > 0$.

$$t^2y'' + 4ty' + 2y = 0$$

Solution

Because the ODE is equidimensional, the solution is of the form $y = t^r$. Substitute it into the equation to determine r.

$$\begin{aligned}
t^2y'' + 4ty' + 2y &= 0 \\
t^2\frac{d^2}{dt^2}(t^r) + 4t\frac{d}{dt}(t^r) + 2(t^r) &= 0 \\
t^2r(r-1)t^{r-2} + 4trt^{r-1} + 2t^r &= 0 \\
r(r-1)t^r + 4rt^r + 2t^r &= 0
\end{aligned}$$

Divide both sides by t^r.

$$r(r-1) + 4r + 2 = 0$$

$$r^2 + 3r + 2 = 0$$

$$(r+2)(r+1) = 0$$

$$r = -2 \quad \text{or} \quad r = -1$$

Therefore, t^{-2} and t^{-1} are two solutions to the ODE. The general solution is

$$y(t) = C_1t^{-2} + C_2t^{-1},$$

where C_1 and C_2 are arbitrary constants.