Problem 19

In each of Problems 19 and 20, determine the values of \( r \) for which the given differential equation has solutions of the form \( y = t^r \) for \( t > 0 \).

\[
t^2 y'' + 4ty' + 2y = 0
\]

Solution

Because the ODE is equidimensional, the solution is of the form \( y = t^r \). Substitute it into the equation to determine \( r \).

\[
t^2 y'' + 4ty' + 2y = 0
\]

\[
t^2 \frac{d^2}{dt^2} (t^r) + 4t \frac{d}{dt} (t^r) + 2(t^r) = 0
\]

\[
t^2 r(r-1)t^{r-2} + 4tr^{r-1} + 2t^r = 0
\]

\[
r(r-1)t^r + 4rt^r + 2t^r = 0
\]

Divide both sides by \( t^r \).

\[
r(r-1) + 4r + 2 = 0
\]

\[
r^2 + 3r + 2 = 0
\]

\[
(r + 2)(r + 1) = 0
\]

\[
r = -2 \quad \text{or} \quad r = -1
\]

Therefore, \( t^{-2} \) and \( t^{-1} \) are two solutions to the ODE. The general solution is

\[
y(t) = C_1 t^{-2} + C_2 t^{-1},
\]

where \( C_1 \) and \( C_2 \) are arbitrary constants.