Problem 30

Another way to derive the pendulum equation (12) is based on the principle of conservation of energy.

(a) Show that the kinetic energy T of the pendulum in motion is

$$T = \frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2.$$

(b) Show that the potential energy V of the pendulum, relative to its rest position, is

$$V = mgL(1 - \cos\theta).$$

(c) By the principle of conservation of energy, the total energy $E = T + V$ is constant. Calculate dE/dt, set it equal to zero, and show that the resulting equation reduces to Eq. (12).

Solution

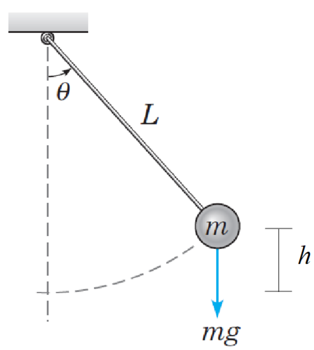

FIGURE 1.3.1 An oscillating pendulum.

Part (a)

The kinetic energy is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(L\frac{d\theta}{dt}\right)^2 = \frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2.$$

Part (b)

The potential energy is

$$V = mgh = mg(L - L\cos\theta) = mgL(1 - \cos\theta).$$

Part (c)

By conservation of energy,

$$E = T + V = \text{constant}.$$

Differentiate all sides with respect to t.

$$\frac{dE}{dt} = \frac{dT}{dt} + \frac{dV}{dt} = 0$$

$$\frac{d}{dt}\left[\frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2\right] + \frac{d}{dt}[mgL(1-\cos\theta)] = 0$$

$$\frac{1}{2}mL^2 \cdot 2\left(\frac{d\theta}{dt}\right)\frac{d^2\theta}{dt^2} + mgL(\sin\theta)\frac{d\theta}{dt} = 0$$

Divide both sides by $mL^2 d\theta/dt$.

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0 \tag{12}$$

This is equation (12) in the textbook.