Problem 30

Another way to derive the pendulum equation (12) is based on the principle of conservation of energy.

(a) Show that the kinetic energy \( T \) of the pendulum in motion is

\[
T = \frac{1}{2} mL^2 \left( \frac{d\theta}{dt} \right)^2.
\]

(b) Show that the potential energy \( V \) of the pendulum, relative to its rest position, is

\[
V = mgL(1 - \cos \theta).
\]

(c) By the principle of conservation of energy, the total energy \( E = T + V \) is constant. Calculate \( dE/dt \), set it equal to zero, and show that the resulting equation reduces to Eq. (12).

Solution

![Diagram of a pendulum](image)

**FIGURE 1.3.1** An oscillating pendulum.

**Part (a)**

The kinetic energy is

\[
T = \frac{1}{2} mv^2 = \frac{1}{2} m \left( L \frac{d\theta}{dt} \right)^2 = \frac{1}{2} mL^2 \left( \frac{d\theta}{dt} \right)^2.
\]

**Part (b)**

The potential energy is

\[
V = mgh = mg(L - L \cos \theta) = mgL(1 - \cos \theta).
\]

**Part (c)**

By conservation of energy,

\[
E = T + V = \text{constant}.
\]
Differentiate all sides with respect to $t$.

\[
\frac{dE}{dt} = \frac{dT}{dt} + \frac{dV}{dt} = 0
\]

\[
\frac{d}{dt} \left[ \frac{1}{2} mL^2 \left( \frac{d\theta}{dt} \right)^2 \right] + \frac{d}{dt}[mgL(1 - \cos \theta)] = 0
\]

\[
\frac{1}{2} mL^2 \cdot 2 \left( \frac{d\theta}{dt} \right) \frac{d^2\theta}{dt^2} + mgL(\sin \theta) \frac{d\theta}{dt} = 0
\]

Divide both sides by $mL^2 \frac{d\theta}{dt}$.

\[
\frac{d^2\theta}{dt^2} + \frac{g}{L} \sin \theta = 0
\]  \hspace{1cm} (12)

This is equation (12) in the textbook.