Problem 31

A third derivation of the pendulum equation depends on the principle of angular momentum: The rate of change of angular momentum about any point is equal to the net external moment about the same point.

(a) Show that the angular momentum M, or moment of momentum, about the point of support is given by $M = mL^2 d\theta/dt$.

(b) Set dM/dt equal to the moment of the gravitational force, and show that the resulting equation reduces to Eq. (12). Note that positive moments are counterclockwise.

Solution

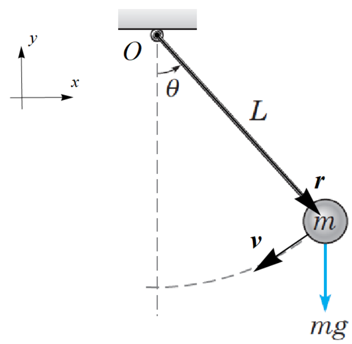

FIGURE 1.3.1 An oscillating pendulum.

Part (a)

Angular momentum is defined as the vector product of the position and momentum vectors.

$$\begin{aligned}
\mathbf{M} &= \mathbf{r} \times m\mathbf{v} \\
&= \mathbf{r} \times m\frac{d\mathbf{r}}{dt} \\
&= \langle L\sin\theta, -L\cos\theta, 0\rangle \times m\frac{d}{dt}\langle L\sin\theta, -L\cos\theta, 0\rangle \\
&= m\langle L\sin\theta, -L\cos\theta, 0\rangle \times \left\langle L\cos\theta\frac{d\theta}{dt}, L\sin\theta\frac{d\theta}{dt}, 0\right\rangle \\
&= mL^2\frac{d\theta}{dt}\langle \sin\theta, -\cos\theta, 0\rangle \times \langle \cos\theta, \sin\theta, 0\rangle \\
&= mL^2\frac{d\theta}{dt}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \sin\theta & -\cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \end{vmatrix} \\
&= mL^2\frac{d\theta}{dt}(\sin^2\theta + \cos^2\theta)\hat{\mathbf{z}} \\
&= mL^2\frac{d\theta}{dt}\hat{\mathbf{z}}
\end{aligned}$$

Part (b)

The rate of change of angular momentum is equal to the moment of the gravitational force.

$$\begin{aligned}
\frac{d\mathbf{M}}{dt} &= \mathbf{r} \times m\mathbf{g} \\
\frac{d}{dt}\left(mL^2 \frac{d\theta}{dt}\hat{\mathbf{z}}\right) &= \langle L\sin\theta, -L\cos\theta, 0\rangle \times \langle 0, -mg, 0\rangle \\
mL^2 \frac{d^2\theta}{dt^2}\hat{\mathbf{z}} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ L\sin\theta & -L\cos\theta & 0 \\ 0 & -mg & 0 \end{vmatrix} \\
mL^2 \frac{d^2\theta}{dt^2}\hat{\mathbf{z}} &= -mgL\sin\theta\hat{\mathbf{z}}
\end{aligned}$$

The components of these vectors are equal.

$$mL^2 \frac{d^2\theta}{dt^2} = -mgL\sin\theta$$

Divide both sides by mL^2

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

and bring both terms to one side.

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0 \tag{12}$$

This is equation (12) in the textbook.