Problem 3

In each of Problems 1 through 12:

(a) Draw a direction field for the given differential equation.

(b) Based on an inspection of the direction field, describe how solutions behave for large $t$.

(c) Find the general solution of the given differential equation, and use it to determine how solutions behave as $t \to \infty$.

\[ y' + y = te^{-t} + 1 \]

Solution

The direction field is a two-dimensional vector field that shows what the direction of the solution is at every point in a region. Every solution to the differential equation is a curve drawn such that the direction field vectors are tangent to it at every point.

\[ \langle dt, dy \rangle = \left\langle 1, \frac{dy}{dt} \right\rangle dt = \left\langle 1, -y + te^{-t} + 1 \right\rangle dt \]

Figure 1: In blue are the direction field vectors and in red are possible solutions to the differential equation, depending what the initial condition is. All solutions appear to converge to $y = 1$ as $t \to \infty$. 

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\[ y' + y = te^{-t} + 1 \]

This is a first-order linear inhomogeneous equation, so it can be solved with an integrating factor.

\[ I(t) = \exp \left[ \int^t ds \right] = e^t \]

Multiply both sides of the ODE by \( I(t) \).

\[ e^t y' + e^t y = t + e^t \]

The left side can be written as \( \frac{d}{dt}(Iy) \) by the product rule.

\[ \frac{d}{dt}(e^t y) = t + e^t \]

Integrate both sides with respect to \( t \).

\[ e^t y = \frac{t^2}{2} + e^t + C \]

Divide both sides by \( e^t \) to solve for \( y(t) \).

\[ y(t) = 1 + e^{-t} \left( \frac{t^2}{2} + C \right) \]

The limit of \( y(t) \) as \( t \to \infty \) is indeed \( y = 1 \).

\[ \lim_{t \to \infty} y(t) = 1 \]