Problem 8

In each of Problems 1 through 12:

(a) Draw a direction field for the given differential equation.

(b) Based on an inspection of the direction field, describe how solutions behave for large t.

(c) Find the general solution of the given differential equation, and use it to determine how solutions behave as $t \to \infty$.

$$(1+t^2)y' + 4ty = (1+t^2)^{-2}$$

Solution

The direction field is a two-dimensional vector field that shows what the direction of the solution is at every point in a region. Every solution to the differential equation is a curve drawn such that the direction field vectors are tangent to it at every point.

$$\langle dt, dy \rangle = \left\langle 1, \frac{dy}{dt} \right\rangle dt = \left\langle 1, \frac{-4ty + (1+t^2)^{-2}}{1+t^2} \right\rangle dt$$

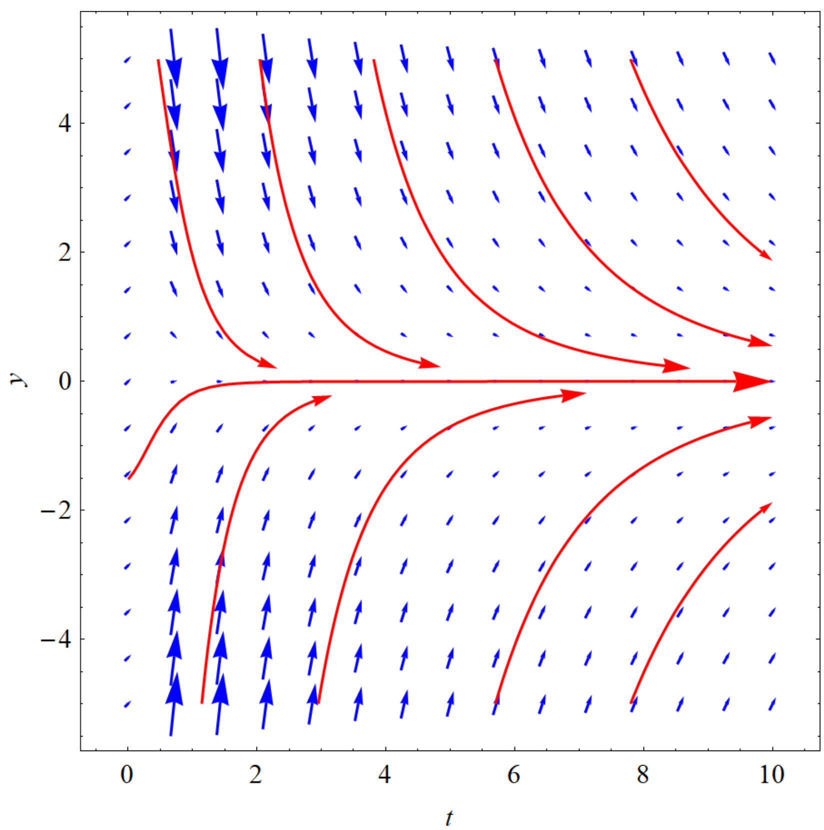

Figure 1: In blue are the direction field vectors and in red are possible solutions to the differential equation, depending what the initial condition is. All solutions appear to converge to $y = 0$ as $t \to \infty$.

$$(1+t^2)y' + 4ty = (1+t^2)^{-2}$$

Divide both sides by $1+t^2$ to make the coefficient of y' 1.

$$y' + \frac{4t}{1+t^2}y = (1+t^2)^{-3}$$

This is a first-order linear inhomogeneous equation, so it can be solved with an integrating factor.

$$I(t) = \exp\left(\int^t \frac{4s}{1+s^2}\,ds\right) = e^{2\ln(1+t^2)} = e^{\ln(1+t^2)^2} = (1+t^2)^2$$

Multiply both sides of the ODE by $I(t)$.

$$(1+t^2)^2y' + 4t(1+t^2)y = \frac{1}{1+t^2}$$

The left side can be written as $d/dt(Iy)$ by the product rule.

$$\frac{d}{dt}[(1+t^2)^2y] = \frac{1}{1+t^2}$$

Integrate both sides with respect to t.

$$(1+t^2)^2y = \tan^{-1}t + C$$

Divide both sides by $(1+t^2)^2$ to solve for $y(t)$.

$$y(t) = \frac{1}{(1+t^2)^2}(\tan^{-1}t + C)$$

$y(t)$ vanishes as $t \to \infty$.

$$\lim_{t\to\infty} y(t) = 0$$