Problem 9

In each of Problems 1 through 12:

(a) Draw a direction field for the given differential equation.

(b) Based on an inspection of the direction field, describe how solutions behave for large $t$.

(c) Find the general solution of the given differential equation, and use it to determine how solutions behave as $t \to \infty$.

$$2y' + y = 3t$$

Solution

The direction field is a two-dimensional vector field that shows what the direction of the solution is at every point in a region. Every solution to the differential equation is a curve drawn such that the direction field vectors are tangent to it at every point.

$$\langle dt, dy \rangle = \left\langle 1, \frac{dy}{dt} \right\rangle dt = \left\langle 1, \frac{-y + 3t}{2} \right\rangle dt$$

Figure 1: In blue are the direction field vectors and in red are possible solutions to the differential equation, depending what the initial condition is. In green is the asymptotic solution $y = 3(t - 2)$ that all other solutions tend to as $t \to \infty$. 

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\[ 2y' + y = 3t \]

Divide both sides by 2 to make the coefficient of \( y' \) 1.

\[ y' + \frac{y}{2} = \frac{3}{2} t \]

This is a first-order linear inhomogeneous equation, so it can be solved with an integrating factor.

\[
I(t) = \exp\left(\int t \frac{1}{2} ds\right) = e^{t/2}
\]

Multiply both sides of the ODE by \( I(t) \).

\[
e^{t/2}y' + \frac{1}{2} e^{t/2} y = \frac{3}{2} te^{t/2}
\]

The left side can be written as \( d/dt(Iy) \) by the product rule.

\[
\frac{d}{dt}(e^{t/2}y) = \frac{3}{2} te^{t/2}
\]

Integrate both sides with respect to \( t \).

\[
e^{t/2}y = \int^t 3 \frac{3}{2} se^{s/2} ds + C = 3e^{t/2}(t - 2) + C
\]

Divide both sides by \( e^{t/2} \) to solve for \( y(t) \).

\[
y(t) = 3(t - 2) + Ce^{-t/2}
\]

\( y(t) \) tends to \( 3(t - 2) \) asymptotically as \( t \to \infty \).