**Problem 10**

In each of Problems 1 through 12:

(a) Draw a direction field for the given differential equation.

(b) Based on an inspection of the direction field, describe how solutions behave for large $t$.

(c) Find the general solution of the given differential equation, and use it to determine how solutions behave as $t \to \infty$.

$$ty' - y = t^2 e^{-t}, \quad t > 0$$

**Solution**

The direction field is a two-dimensional vector field that shows what the direction of the solution is at every point in a region. Every solution to the differential equation is a curve drawn such that the direction field vectors are tangent to it at every point.

$$\langle dt, dy \rangle = \left\langle 1, \frac{dy}{dt} \right\rangle dt = \left\langle 1, \frac{y + t^2 e^{-t}}{t} \right\rangle dt$$

![Direction Field Diagram](image)

**Figure 1:** In blue are the direction field vectors and in red are possible solutions to the differential equation, depending what the initial condition is. All solutions appear to diverge as $t \to \infty$.  

www.stemjock.com
\[ ty' - y = t^2 e^{-t} \]

Divide both sides by \( t \) to make the coefficient of \( y' \) 1.

\[ y' - \frac{y}{t} = te^{-t} \]

This is a first-order linear inhomogeneous equation, so it can be solved with an integrating factor.

\[ I(t) = \exp \left( \int t^0 \frac{1}{s} \, ds \right) = e^{-\ln t} = e^{\ln t^{-1}} = \frac{1}{t} \]

Multiply both sides of the ODE by \( I(t) \).

\[ \frac{y'}{t} - \frac{y}{t^2} = e^{-t} \]

The left side can be written as \( d/dt(Iy) \) by the product rule.

\[ \frac{d}{dt} \left( \frac{y}{t} \right) = e^{-t} \]

Integrate both sides with respect to \( t \).

\[ \frac{y}{t} = -e^{-t} + C \]

Multiply both sides by \( t \) to solve for \( y(t) \).

\[ y(t) = t(-e^{-t} + C) \]

\( y(t) \) is asymptotic to \( Ct \) as \( t \to \infty \).