

Problem 11

In each of Problems 1 through 12:

- Draw a direction field for the given differential equation.
- Based on an inspection of the direction field, describe how solutions behave for large t .
- Find the general solution of the given differential equation, and use it to determine how solutions behave as $t \rightarrow \infty$.

$$y' + y = 5 \sin 2t$$

Solution

The direction field is a two-dimensional vector field that shows what the direction of the solution is at every point in a region. Every solution to the differential equation is a curve drawn such that the direction field vectors are tangent to it at every point.

$$\langle dt, dy \rangle = \left\langle 1, \frac{dy}{dt} \right\rangle dt = \langle 1, -y + 5 \sin 2t \rangle dt$$

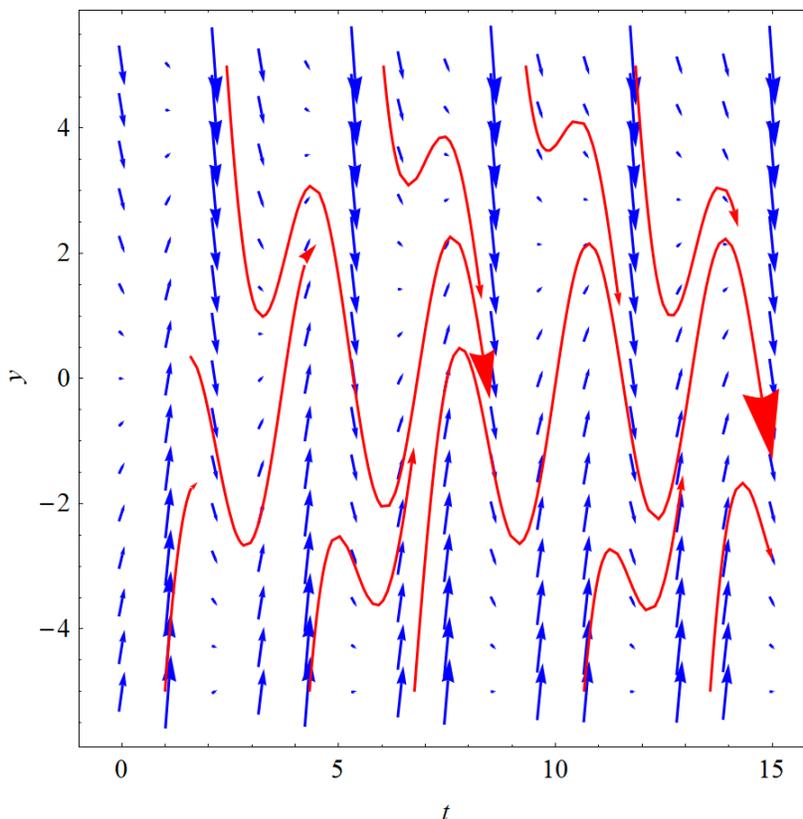


Figure 1: In blue are the direction field vectors and in red are possible solutions to the differential equation, depending what the initial condition is. All solutions appear to tend to a sinusoidal function as $t \rightarrow \infty$.

$$y' + y = 5 \sin 2t$$

This is a first-order linear inhomogeneous equation, so it can be solved with an integrating factor.

$$I(t) = \exp\left(\int^t ds\right) = e^t$$

Multiply both sides of the ODE by $I(t)$.

$$e^t y' + e^t y = 5e^t \sin 2t$$

The left side can be written as $d/dt(Iy)$ by the product rule.

$$\frac{d}{dt}(e^t y) = 5e^t \sin 2t$$

Integrate both sides with respect to t .

$$\begin{aligned} e^t y &= \int^t 5e^s \sin 2s \, ds + C \\ &= e^t(\sin 2t - 2 \cos 2t) + C \end{aligned}$$

Divide both sides by e^t to solve for $y(t)$.

$$y(t) = \sin 2t - 2 \cos 2t + Ce^{-t}$$

$y(t)$ is asymptotic to $\sin 2t - 2 \cos 2t$ as $t \rightarrow \infty$.