Problem 13

In each of Problems 13 through 20, find the solution of the given initial value problem.

\[ y' - y = 2te^{2t}, \quad y(0) = 1 \]

Solution\(^1\)

This is a first-order linear inhomogeneous equation, so it can be solved with an integrating factor.

\[ I(t) = \exp \left[ \int^t (-1) \, ds \right] = e^{-t} \]

Multiply both sides of the ODE by \(I(t)\).

\[ e^{-t}y' - e^{-t}y = 2te^t \]

The left side can be written as \(d/dt(Iy)\) by the product rule.

\[ \frac{d}{dt}(e^{-t}y) = 2te^t \]

Integrate both sides with respect to \(t\).

\[
e^{-t}y = \int^t 2se^s \, ds + C
= 2 \int^t s \frac{d}{ds}(e^s) \, ds + C
= 2 \left[ se^s \right]^t_0 - \int^t \frac{d}{ds}(s)e^s \, ds + C
= 2 \left( te^t - \int^t e^s \, ds \right) + C
= 2(te^t - e^t) + C
= 2(t - 1)e^t + C
\]

Multiply both sides by \(e^t\) to solve for \(y(t)\).

\[ y(t) = 2(t - 1)e^{2t} + Ce^t \]

Apply the initial condition \(y(0) = 1\) here to determine \(C\).

\[ y(0) = 2(-1)e^0 + Ce^0 = 1 \quad \rightarrow \quad C = 1 + 2 = 3 \]

Therefore,

\[ y(t) = 2(t - 1)e^{2t} + 3e^t. \]

\(^1\)Special thanks to A. Park for the correction in the problem statement.

www.stemjock.com