Problem 13

In each of Problems 13 through 20, find the solution of the given initial value problem.

$$y' - y = 2te^{2t}, \qquad y(0) = 1$$

Solution[1]

This is a first-order linear inhomogeneous equation, so it can be solved with an integrating factor.

$$I(t) = \exp\left[\int^t (-1)\,ds\right] = e^{-t}$$

Multiply both sides of the ODE by $I(t)$.

$$e^{-t}y' - e^{-t}y = 2te^t$$

The left side can be written as $d/dt(Iy)$ by the product rule.

$$\frac{d}{dt}(e^{-t}y) = 2te^t$$

Integrate both sides with respect to t.

$$\begin{aligned} e^{-t}y &= \int^t 2se^s\,ds + C \\ &= 2\int^t s\frac{d}{ds}(e^s)\,ds + C \\ &= 2\left[se^s\Big|^t - \int^t \frac{d}{ds}(s)e^s\,ds\right] + C \\ &= 2\left(te^t - \int^t e^s\,ds\right) + C \\ &= 2(te^t - e^t) + C \\ &= 2(t-1)e^t + C \end{aligned}$$

Multiply both sides by e^t to solve for $y(t)$.

$$y(t) = 2(t-1)e^{2t} + Ce^t$$

Apply the initial condition $y(0) = 1$ here to determine C.

$$y(0) = 2(-1)e^0 + Ce^0 = 1 \quad \rightarrow \quad C = 1 + 2 = 3$$

Therefore,

$$y(t) = 2(t-1)e^{2t} + 3e^t.$$

[1]Special thanks to A. Park for the correction in the problem statement.