Problem 18

In each of Problems 13 through 20, find the solution of the given initial value problem.

$$ty' + 2y = \sin t, \qquad y(\pi/2) = 1, \quad t > 0$$

Solution

Start by dividing both sides by t to make the coefficient of y' 1.

$$y' + \frac{2}{t}y = \frac{\sin t}{t}$$

This is a first-order linear inhomogeneous equation, so it can be solved with an integrating factor.

$$I(t) = \exp\left(\int^t \frac{2}{s}\,ds\right) = e^{2\ln t} = e^{\ln t^2} = t^2$$

Multiply both sides of the ODE by $I(t)$.

$$t^2 y' + 2ty = t\sin t$$

The left side can be written as $d/dt(Iy)$ by the product rule.

$$\frac{d}{dt}(t^2 y) = t\sin t$$

Integrate both sides with respect to t.

$$\begin{aligned} t^2 y &= \int^t s\sin s\,ds + C \\ &= \sin t - t\cos t + C \end{aligned}$$

Divide both sides by t^2 to solve for $y(t)$.

$$y(t) = \frac{\sin t - t\cos t + C}{t^2}$$

Apply the initial condition $y(\pi/2) = 1$ here to determine C.

$$y\left(\frac{\pi}{2}\right) = \frac{1 - 0 + C}{\frac{\pi^2}{4}} = 1 \quad \rightarrow \quad C = \frac{\pi^2}{4} - 1$$

Therefore,

$$y(t) = \frac{\sin t - t\cos t + \frac{\pi^2}{4} - 1}{t^2}.$$