Problem 20

In each of Problems 13 through 20, find the solution of the given initial value problem.

$$ty' + (t+1)y = t, \qquad y(\ln 2) = 1, \quad t > 0$$

Solution

Start by dividing both sides by t to make the coefficient of y' 1.

$$y' + \left(1 + \frac{1}{t}\right) y = 1$$

This is a first-order linear inhomogeneous equation, so it can be solved with an integrating factor.

$$I(t) = \exp\left[\int^t \left(1 + \frac{1}{s}\right) ds\right] = e^{t+\ln t} = te^t$$

Multiply both sides of the ODE by $I(t)$.

$$te^t y' + (t+1)e^t y = te^t$$

The left side can be written as $d/dt(Iy)$ by the product rule.

$$\frac{d}{dt}(te^t y) = te^t$$

Integrate both sides with respect to t.

$$\begin{aligned} te^t y &= \int^t se^s\, ds + C \\ &= e^t(t-1) + C \end{aligned}$$

Divide both sides by te^t to solve for $y(t)$.

$$y(t) = \frac{t-1}{t} + \frac{C}{te^t}$$

Apply the initial condition $y(\ln 2) = 1$ here to determine C.

$$y(\ln 2) = \frac{\ln 2 - 1}{\ln 2} + \frac{C}{(\ln 2)(2)} = 1 \quad \rightarrow \quad C = 2$$

Therefore,

$$\begin{aligned} y(t) &= \frac{t-1}{t} + \frac{2}{te^t} \\ &= \frac{t - 1 + 2e^{-t}}{t}. \end{aligned}$$