Problem 20

In each of Problems 13 through 20, find the solution of the given initial value problem.

\[ ty' + (t + 1)y = t, \quad y(\ln 2) = 1, \quad t > 0 \]

Solution

Start by dividing both sides by \( t \) to make the coefficient of \( y' \) 1.

\[ y' + \left(1 + \frac{1}{t}\right)y = 1 \]

This is a first-order linear inhomogeneous equation, so it can be solved with an integrating factor.

\[ I(t) = \exp \left[ \int^t s \, ds \right] = e^{t + \ln t} = te^t \]

Multiply both sides of the ODE by \( I(t) \).

\[ te^t y' + (t + 1)e^t y = te^t \]

The left side can be written as \( d/dt(Iy) \) by the product rule.

\[ \frac{d}{dt}(te^t y) = te^t \]

Integrate both sides with respect to \( t \).

\[ te^t y = \int^t se^s \, ds + C = e^t(t - 1) + C \]

Divide both sides by \( te^t \) to solve for \( y(t) \).

\[ y(t) = \frac{t - 1}{t} + \frac{C}{te^t} \]

Apply the initial condition \( y(\ln 2) = 1 \) here to determine \( C \).

\[ y(\ln 2) = \frac{\ln 2 - 1}{\ln 2} + \frac{C}{(\ln 2)(2)} = 1 \quad \rightarrow \quad C = 2 \]

Therefore,

\[ y(t) = \frac{t - 1}{t} + \frac{2}{te^t} = \frac{t - 1 + 2e^{-t}}{t}. \]