Problem 21

In each of Problems 21 through 23:

(a) Draw a direction field for the given differential equation. How do solutions appear to behave as t becomes large? Does the behavior depend on the choice of the initial value a? Let a_0 be the value of a for which the transition from one type of behavior to another occurs. Estimate the value of a_0.

(b) Solve the initial value problem and find the critical value a_0 exactly.

(c) Describe the behavior of the solution corresponding to the initial value a_0.

$$y' - \frac{1}{2}y = 2\cos t, \qquad y(0) = a$$

Solution

Part (a)

The direction field is a two-dimensional vector field that shows what the direction of the solution is at every point in a region. Every solution to the differential equation is a curve drawn such that the direction field vectors are tangent to it at every point.

$$\langle dt, dy \rangle = \left\langle 1, \frac{dy}{dt} \right\rangle dt = \left\langle 1, \frac{1}{2}y + 2\cos t \right\rangle dt$$

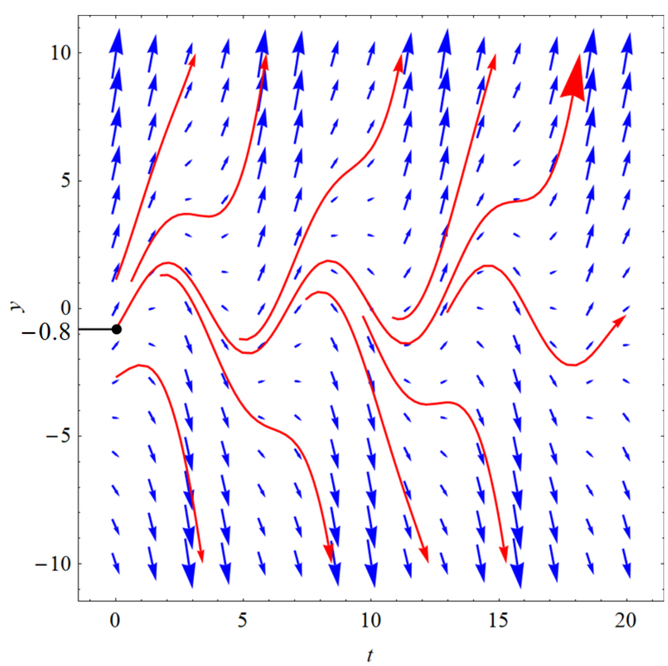

Figure 1: In blue are the direction field vectors and in red are possible solutions to the differential equation, depending what the initial condition is. The solution diverges to ∞ or $-\infty$, depending whether the initial condition is above or below a sinusoidal curve, respectively. $a_0 \approx -0.8$

Part (b)

$$y' - \frac{1}{2}y = 2\cos t$$

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left[\int^t \left(-\frac{1}{2}\right) ds\right] = e^{-t/2}$$

Proceed with the multiplication.

$$e^{-t/2}y' - \frac{1}{2}e^{-t/2}y = 2e^{-t/2}\cos t$$

The left side can be written as $d/dt(Iy)$ using the product rule.

$$\frac{d}{dt}(e^{-t/2}y) = 2e^{-t/2}\cos t$$

Integrate both sides with respect to t.

$$e^{-t/2}y = \int^t 2e^{-s/2}\cos s\,ds + C \tag{1}$$

The integral can be evaluated using integration by parts twice.

$$\begin{aligned}
\int^t 2e^{-s/2}\cos s\,ds &= \int^t 2e^{-s/2}\frac{d}{ds}(\sin s)\,ds \\
&= 2e^{-t/2}\sin t - \int^t 2\left(-\frac{1}{2}\right)e^{-s/2}\sin s\,ds \\
&= 2e^{-t/2}\sin t + \int^t e^{-s/2}\sin s\,ds \\
&= 2e^{-t/2}\sin t + \int^t e^{-s/2}\frac{d}{ds}(-\cos s)\,ds \\
&= 2e^{-t/2}\sin t + \left[e^{-t/2}(-\cos t) - \int^t \left(-\frac{1}{2}\right)e^{-s/2}(-\cos s)\,ds\right] \\
&= 2e^{-t/2}\sin t - e^{-t/2}\cos t - \frac{1}{2}\int^t e^{-s/2}\cos s\,ds
\end{aligned}$$

Solve the equation for the desired integral.

$$\frac{5}{2}\int^t e^{-s/2}\cos s\,ds = 2e^{-t/2}\sin t - e^{-t/2}\cos t \quad \rightarrow \quad 2\int^t e^{-s/2}\cos s\,ds = \frac{4}{5}e^{-t/2}(2\sin t - \cos t)$$

Continue now from equation (1).

$$e^{-t/2}y = \frac{4}{5}e^{-t/2}(2\sin t - \cos t) + C$$

Multiply both sides by $e^{t/2}$ to obtain the general solution for y.

$$y(t) = \frac{4}{5}(2\sin t - \cos t) + Ce^{t/2}$$

Apply the initial condition $y(0) = a$ now to determine C.

$$y(0) = \frac{4}{5}(-1) + C = a \quad \rightarrow \quad C = a + \frac{4}{5}$$

Therefore, the solution to the initial value problem is

$$y(t) = \frac{4}{5}(2\sin t - \cos t) + \left(a + \frac{4}{5}\right)e^{t/2}.$$

For values of a below $-4/5$, the solution diverges to $-\infty$ as $t \to \infty$, and for values of a above $-4/5$, the solution diverges to ∞ as $t \to \infty$. Therefore,

$$a_0 = -\frac{4}{5}.$$

Part (c)

If $a = -4/5$, then the solution to the initial value problem reduces to

$$y(t) = \frac{4}{5}(2\sin t - \cos t),$$

which is bounded as $t \to \infty$. This is the sinusoidal curve in Figure 1 above and below which other solutions diverge. It alone is plotted as a function of t below.