Problem 32

Show that all solutions of $2y' + ty = 2$ [Eq. (41) of the text] approach a limit as $t \to \infty$, and find the limiting value.
Hint: Consider the general solution, Eq. (47), and use L'Hôpital's rule on the first term.

Solution

Divide both sides of the ODE by 2 so that the coefficient of y' is 1.

$$y' + \frac{t}{2}y = 1$$

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left[\int^t \left(\frac{s}{2}\right) ds\right] = e^{t^2/4}$$

Proceed with the multiplication.

$$e^{t^2/4}y' + \frac{t}{2}e^{t^2/4}y = e^{t^2/4}$$

The left side can be written as $d/dt(Iy)$ using the product rule.

$$\frac{d}{dt}(e^{t^2/4}y) = e^{t^2/4}$$

Integrate both sides with respect to t.

$$e^{t^2/4}y = \int^t e^{s^2/4}\, ds + C$$

Divide both sides by $e^{t^2/4}$ to obtain the general solution for y.

$$y(t) = \frac{\int^t e^{s^2/4}\, ds}{e^{t^2/4}} + \frac{C}{e^{t^2/4}}$$

Take the limit of both sides as $t \to \infty$.

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{\int_0^t e^{s^2/4}\, ds}{e^{t^2/4}} + \underbrace{\lim_{t\to\infty} \frac{C}{e^{t^2/4}}}_{=0}$$

The remaining term is an indeterminate form (∞/∞), so l'Hôpital's rule can be applied to evaluate the limit.

$$\lim_{t\to\infty} y(t) \overset{\frac{\infty}{\infty}}{\underset{\text{H}}{=}} \lim_{t\to\infty} \frac{\frac{d}{dt}\int_0^t e^{s^2/4}\, ds}{\frac{d}{dt}e^{t^2/4}} = \lim_{t\to\infty} \frac{e^{t^2/4}}{\left(\frac{2t}{4}\right)e^{t^2/4}} = \lim_{t\to\infty} \frac{2}{t} = 0$$

We conclude that all solutions of this ODE tend to zero as $t \to \infty$.