Problem 37

In each of Problems 34 through 37, construct a first order linear differential equation whose solutions have the required behavior as $t \to \infty$. Then solve your equation and confirm that the solutions do indeed have the specified property.

All solutions approach the curve $y = 4 - t^2$ as $t \to \infty$.

Solution

The rate of change of y will become $-2t$ as t gets big enough, so we choose

$$\begin{aligned} y' + y &= (-2t) + (4 - t^2) \\ &= 4 - 2t - t^2. \end{aligned}$$

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^t 1\, ds\right) = e^t$$

Proceed with the multiplication.

$$e^t y' + e^t y = e^t(4 - 2t - t^2)$$

The left side can be written as $d/dt(Iy)$ using the product rule.

$$\frac{d}{dt}(e^t y) = e^t(4 - 2t - t^2)$$

Integrate both sides with respect to t.

$$\begin{aligned} e^t y &= \int^t e^s(4 - 2s - s^2)\, ds + C \\ &= \int^t 4e^s\, ds - \int^t 2se^s\, ds - \int^t s^2 e^s\, ds + C \\ &= 4e^t - \int^t 2se^s\, ds - \int^t s^2 \frac{d}{ds}(e^s)\, ds + C \\ &= 4e^t - \int^t 2se^s\, ds - \left[t^2(e^t) - \int^t (2s)(e^s)\, ds\right] + C \\ &= 4e^t - \cancel{\int^t 2se^s\, ds} - t^2 e^t + \cancel{\int^t 2se^s\, ds} + C \\ &= e^t(4 - t^2) + C \end{aligned}$$

Divide both sides by e^t to obtain the general solution for y.

$$y(t) = 4 - t^2 + \frac{C}{e^t}$$

Take the limit of both sides as $t \to \infty$.

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}(4 - t^2) + \underbrace{\lim_{t\to\infty} \frac{C}{e^t}}_{=\,0}$$

Therefore, all solutions approach the curve $y = 4 - t^2$ as $t \to \infty$.