Problem 40

In each of Problems 39 through 42, use the method of Problem 38 to solve the given differential equation.

$$y' + (1/t)y = 3\cos 2t, \qquad t > 0$$

Solution

The method of variation of parameters will be used here. Start by solving the associated homogeneous equation.

$$Y' + \frac{1}{t}Y = 0$$

Divide both sides by Y

$$\frac{Y'}{Y} + \frac{1}{t} = 0$$

and bring $(1/t)$ to the right side.

$$\frac{Y'}{Y} = -\frac{1}{t}$$

The left side can be written as $d/dt(\ln|Y|)$ by the chain rule. The absolute value sign has been included because the argument of the logarithm cannot be negative.

$$\frac{d}{dt}(\ln|Y|) = -\frac{1}{t}$$

Integrate both sides with respect to t.

$$\ln|Y| = -\ln t + C$$

Exponentiate both sides.

$$\begin{aligned}|Y| &= e^{-\ln t + C}\\ &= e^C e^{\ln t^{-1}}\\ &= e^C t^{-1}\end{aligned}$$

Introduce $\pm$ on the right side to remove the absolute value sign.

$$Y(t) = \pm e^C t^{-1}$$

Therefore, using a new constant A for $\pm e^C$,

$$Y(t) = At^{-1}.$$

The solution for y is obtained by allowing the parameter A to vary.

$$y(t) = A(t)t^{-1} \tag{1}$$

Substitute this formula for y into the original ODE to obtain an equation for $A(t)$.

$$y' + \frac{1}{t}y = 3\cos 2t \quad \rightarrow \quad [A(t)t^{-1}]' + \frac{1}{t}[A(t)t^{-1}] = 3\cos 2t$$

Use the product rule to simplify the left side.

$$A'(t)t^{-1} + \cancel{A(t)(-t^{-2})} + \cancel{A(t)t^{-2}} = 3\cos 2t$$

Multiply both sides by t.

$$A'(t) = 3t\cos 2t$$

Integrate both sides with respect to t.

$$\begin{aligned}
A(t) &= \int^t 3s\cos 2s\,ds + C_1 \\
&= 3\int^t s\frac{d}{ds}\left(\frac{1}{2}\sin 2s\right)ds + C_1 \\
&= 3\left[t\left(\frac{1}{2}\sin 2t\right) - \int^t (1)\left(\frac{1}{2}\sin 2s\right)ds\right] + C_1 \\
&= 3\left(\frac{1}{2}t\sin 2t - \frac{1}{2}\int^t \sin 2s\,ds\right) + C_1 \\
&= 3\left[\frac{1}{2}t\sin 2t - \frac{1}{2}\left(-\frac{1}{2}\cos 2t\right)\right] + C_1 \\
&= \frac{3}{4}(2t\sin 2t + \cos 2t) + C_1
\end{aligned}$$

Substitute this result into equation (1) to obtain the general solution for y.

$$y(t) = \left[\frac{3}{4}(2t\sin 2t + \cos 2t) + C_1\right]t^{-1}$$