**Problem 40**

In each of Problems 39 through 42, use the method of Problem 38 to solve the given differential equation.

\[ y' + \frac{1}{t}y = 3 \cos 2t, \quad t > 0 \]

**Solution**

The method of variation of parameters will be used here. Start by solving the associated homogeneous equation.

\[ Y' + \frac{1}{t}Y = 0 \]

Divide both sides by \( Y \)

\[ \frac{Y'}{Y} + \frac{1}{t} = 0 \]

and bring \((1/t)\) to the right side.

\[ \frac{Y'}{Y} = -\frac{1}{t} \]

The left side can be written as \( d/dt(\ln |Y|) \) by the chain rule. The absolute value sign has been included because the argument of the logarithm cannot be negative.

\[ \frac{d}{dt}(\ln |Y|) = -\frac{1}{t} \]

Integrate both sides with respect to \( t \).

\[ \ln |Y| = -\ln t + C \]

Exponentiate both sides.

\[ |Y| = e^{-\ln t+C} \]

\[ = e^C e^{\ln t^{-1}} \]

\[ = e^C t^{-1} \]

Introduce \( \pm \) on the right side to remove the absolute value sign.

\[ Y(t) = \pm e^C t^{-1} \]

Therefore, using a new constant \( A \) for \( \pm e^C \),

\[ Y(t) = At^{-1} \]

The solution for \( y \) is obtained by allowing the parameter \( A \) to vary.

\[ y(t) = A(t)t^{-1} \quad (1) \]

Substitute this formula for \( y \) into the original ODE to obtain an equation for \( A(t) \).

\[ y' + \frac{1}{t}y = 3 \cos 2t \quad \rightarrow \quad [A(t)t^{-1}]' + \frac{1}{t}[A(t)t^{-1}] = 3 \cos 2t \]
Use the product rule to simplify the left side.

\[ A'(t) t^{-1} + A(t)(-t^{-2}) + A(t) t^{-2} = 3 \cos 2t \]

Multiply both sides by \( t \).

\[ A'(t) = 3t \cos 2t \]

Integrate both sides with respect to \( t \).

\[
A(t) = \int t \, 3 \cos 2s \, ds + C_1
\]

\[
= 3 \int t \, \frac{d}{ds} \left( \frac{1}{2} \sin 2s \right) \, ds + C_1
\]

\[
= 3 \left[ t \left( \frac{1}{2} \sin 2t \right) - \int (1) \left( \frac{1}{2} \sin 2s \right) \, ds \right] + C_1
\]

\[
= 3 \left( \frac{1}{2} t \sin 2t - \frac{1}{2} \int \sin 2s \, ds \right) + C_1
\]

\[
= 3 \left( \frac{1}{2} t \sin 2t - \frac{1}{2} \left( -\frac{1}{2} \cos 2t \right) \right) + C_1
\]

\[
= \frac{3}{4} (2t \sin 2t + \cos 2t) + C_1
\]

Substitute this result into equation (1) to obtain the general solution for \( y \).

\[
y(t) = \left[ \frac{3}{4} (2t \sin 2t + \cos 2t) + C_1 \right] t^{-1}
\]