Problem 42

In each of Problems 39 through 42, use the method of Problem 38 to solve the given differential equation.

$$2y' + y = 3t^2$$

Solution

The method of variation of parameters will be used here. Start by solving the associated homogeneous equation.

$$2Y' + Y = 0$$

Divide both sides by $2Y$

$$\frac{Y'}{Y} + \frac{1}{2} = 0$$

and bring (1/2) to the right side.

$$\frac{Y'}{Y} = -\frac{1}{2}$$

The left side can be written as $d/dt(\ln|Y|)$ by the chain rule. The absolute value sign has been included because the argument of the logarithm cannot be negative.

$$\frac{d}{dt}(\ln|Y|) = -\frac{1}{2}$$

Integrate both sides with respect to t.

$$\ln|Y| = -\frac{1}{2}t + C$$

Exponentiate both sides.

$$\begin{aligned} |Y| &= e^{-t/2+C} \\ &= e^C e^{-t/2} \end{aligned}$$

Introduce $\pm$ on the right side to remove the absolute value sign.

$$Y(t) = \pm e^C e^{-t/2}$$

Therefore, using a new constant A for $\pm e^C$,

$$Y(t) = Ae^{-t/2}.$$

The solution for y is obtained by allowing the parameter A to vary.

$$y(t) = A(t)e^{-t/2} \tag{1}$$

Substitute this formula for y into the original ODE to obtain an equation for $A(t)$.

$$2y' + y = 3t^2 \quad \rightarrow \quad 2[A(t)e^{-t/2}]' + [A(t)e^{-t/2}] = 3t^2$$

Use the product rule to simplify the left side.

$$2\left[A'(t)e^{-t/2} + A(t)\left(-\frac{1}{2}\right)e^{-t/2}\right] + A(t)e^{-t/2} = 3t^2$$

$$2A'(t)e^{-t/2} - \cancel{A(t)e^{-t/2}} + \cancel{A(t)e^{-t/2}} = 3t^2$$

Multiply both sides by $e^{t/2}/2$.

$$A'(t) = \frac{3}{2}t^2 e^{t/2}$$

Integrate both sides with respect to t.

$$\begin{aligned}
A(t) &= \int^t \frac{3}{2}s^2 e^{s/2}\,ds + C_1 \\
&= \frac{3}{2}\int^t s^2 \frac{d}{ds}(2e^{s/2})\,ds + C_1 \\
&= \frac{3}{2}\left[t^2(2e^{t/2}) - \int^t (2s)(2e^{s/2})\,ds\right] + C_1 \\
&= \frac{3}{2}\left(2t^2e^{t/2} - 4\int^t se^{s/2}\,ds\right) + C_1 \\
&= 3t^2e^{t/2} - 6\int^t se^{s/2}\,ds + C_1 \\
&= 3t^2e^{t/2} - 6\int^t s\frac{d}{ds}(2e^{s/2})\,ds + C_1 \\
&= 3t^2e^{t/2} - 6\left[t(2e^{t/2}) - \int^t (1)(2e^{s/2})\,ds\right] + C_1 \\
&= 3t^2e^{t/2} - 12te^{t/2} + 12\int^t e^{s/2}\,ds + C_1 \\
&= 3t^2e^{t/2} - 12te^{t/2} + 12(2)e^{t/2} + C_1 \\
&= 3e^{t/2}(t^2 - 4t + 8) + C_1
\end{aligned}$$

Substitute this result into equation (1) to obtain the general solution for y.

$$y(t) = [3e^{t/2}(t^2 - 4t + 8) + C_1]e^{-t/2}$$

Therefore,

$$y(t) = 3(t^2 - 4t + 8) + C_1e^{-t/2}.$$