Problem 2

In each of Problems 1 through 8, solve the given differential equation.

\[ y' = \frac{x^2}{y(1 + x^3)} \]

Solution

This ODE is separable because it is of the form \( y' = f(x)g(y) \), so it can be solved by separating variables.

\[ \frac{dy}{dx} = \frac{x^2}{y(1 + x^3)} \]

Bring the terms with \( y \) to the left and bring the terms with \( x \) to the right.

\[ y \, dy = \frac{x^2}{1 + x^3} \, dx \]

Integrate both sides.

\[ \int y \, dy = \int \frac{x^2}{1 + x^3} \, dx \quad (1) \]

Use a substitution to evaluate the integral on the right.

\[ u = 1 + x^3 \]
\[ du = 3x^2 \, dx \quad \rightarrow \quad \frac{du}{3} = x^2 \, dx \]

Equation (1) becomes

\[ \frac{y^2}{2} = \int \frac{1}{u} \, \frac{du}{3} \]
\[ = \frac{1}{3} \ln |u| + C \]
\[ = \frac{1}{3} \ln |1 + x^3| + C. \]

Now solve for \( y \).

\[ y^2 = \frac{2}{3} \ln |1 + x^3| + 2C \]
\[ y(x) = \pm \sqrt{\frac{2}{3} \ln |1 + x^3| + 2C} \]

Therefore, using a new constant \( C_1 \) for \( 2C \),

\[ y(x) = \pm \sqrt{\frac{2}{3} \ln |1 + x^3| + C_1}. \]