Problem 6

In each of Problems 1 through 8, solve the given differential equation.

$$xy' = (1 - y^2)^{1/2}$$

Solution

Divide both sides by x.

$$y' = \frac{(1 - y^2)^{1/2}}{x}$$

This ODE is separable because it is of the form $y' = f(x)g(y)$, so it can be solved by separating variables.

$$\frac{dy}{dx} = \frac{(1 - y^2)^{1/2}}{x}$$

Bring the terms with y to the left and bring the terms with x to the right.

$$\frac{dy}{(1 - y^2)^{1/2}} = \frac{dx}{x}$$

Integrate both sides.

$$\int \frac{dy}{(1 - y^2)^{1/2}} = \int \frac{dx}{x} \tag{1}$$

Make the following substitution for the integral on the left.

$$\begin{aligned} y &= \sin\theta \quad \rightarrow \quad \theta = \sin^{-1} y \\ dy &= \cos\theta \, d\theta \end{aligned}$$

Equation (1) becomes

$$\int \frac{\cos\theta \, d\theta}{(1 - \sin^2\theta)^{1/2}} = \ln|x| + C$$

$$\int \frac{\cos\theta \, d\theta}{(\cos^2\theta)^{1/2}} = \ln|x| + C$$

$$\int d\theta = \ln|x| + C$$

$$\theta = \ln|x| + C$$

$$\sin^{-1} y = \ln|x| + C.$$

Therefore, taking the sine of both sides,

$$y(x) = \sin(\ln|x| + C).$$