Problem 21

Solve the initial value problem

$$y' = (1 + 3x^2)/(3y^2 - 6y), \qquad y(0) = 1$$

and determine the interval in which the solution is valid.
Hint: To find the interval of definition, look for points where the integral curve has a vertical tangent.

Solution

This ODE is separable because it is of the form $y' = f(x)g(y)$, so it can be solved by separating variables.

$$\frac{dy}{dx} = \frac{1 + 3x^2}{3y^2 - 6y}$$

Bring the terms with y to the left and bring the terms with x to the right.

$$(3y^2 - 6y)\,dy = (1 + 3x^2)\,dx$$

Integrate both sides.

$$\int (3y^2 - 6y)\,dy = \int (1 + 3x^2)\,dx$$

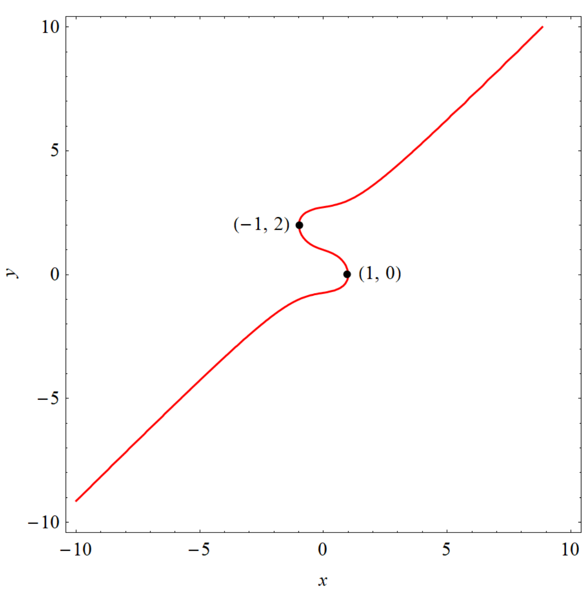
$$y^3 - 3y^2 = x + x^3 + C$$

Use the initial condition now to determine C.

$$1^3 - 3(1)^2 = 0 + 0 + C \quad \to \quad C = -2$$

Therefore, an implicit solution to the ODE is

$$y^3 - 3y^2 = x + x^3 - 2.$$



Factor the denominator in the ODE.

$$\frac{dy}{dx} = \frac{1 + 3x^2}{3y(y-2)}$$

The denominator blows up when $3y(y-2) = 0$, that is,

$$y = 0 \quad \text{or} \quad y = 2.$$

Use the solution to the ODE to find the values of x that correspond to these values of y.

$$y = 0: \quad 0 - 0 = x + x^3 - 2 \quad \rightarrow \quad x = 1$$
$$y = 2: \quad 8 - 12 = x + x^3 - 2 \quad \rightarrow \quad x = -1$$

The solution we found is valid on the part of the curve that passes through the point $(x = 0, y = 1)$ until dy/dx becomes unbounded.

$$-1 < x < 1$$