Problem 24

Solve the initial value problem

\[ y' = \frac{2 - e^x}{3 + 2y}, \quad y(0) = 0 \]

and determine where the solution attains its maximum value.

Solution

This ODE is separable because it is of the form \( y' = f(x)g(y) \), so it can be solved by separating variables.

\[
\frac{dy}{dx} = \frac{2 - e^x}{3 + 2y}
\]

Bring the terms with \( y \) to the left and bring the terms with \( x \) to the right.

\[
(3 + 2y)\, dy = (2 - e^x)\, dx
\]

Integrate both sides.

\[
3y + y^2 = 2x - e^x + C
\]

Apply the initial condition now to determine \( C \).

\[
0 + 0 = 0 - 1 + C \quad \rightarrow \quad C = 1
\]

As a result,

\[
3y + y^2 = 2x - e^x + 1
\]

\[
y^2 + 3y - (2x - e^x + 1) = 0
\]

\[
y(x) = \frac{-3 \pm \sqrt{9 + 4(2x - e^x + 1)}}{2}
\]

We choose the plus sign so that the initial condition is satisfied. Therefore,

\[
y(x) = \frac{-3 + \sqrt{13 + 4(2x - e^x)}}{2}.
\]

Inspecting the ODE, we see that \( dy/dx = 0 \) when \( 2 - e^x = 0 \), or \( x = \ln 2 \approx 0.69 \). Plugging this value into the solution, we get

\[
y(\ln 2) \approx 0.124.
\]

Therefore, the solution attains its maximum value at approximately \((0.69, 0.124)\).