Problem 25

Solve the initial value problem

\[ y' = \frac{2 \cos 2x}{3 + 2y}, \quad y(0) = -1 \]

and determine where the solution attains its maximum value.

Solution

This ODE is separable because it is of the form \( y' = f(x)g(y) \), so it can be solved by separating variables.

\[ \frac{dy}{dx} = \frac{2 \cos 2x}{3 + 2y} \]

Bring the terms with \( y \) to the left and bring the terms with \( x \) to the right.

\[ (3 + 2y) \, dy = 2 \cos 2x \, dx \]

Integrate both sides.

\[ \int (3 + 2y) \, dy = \int 2 \cos 2x \, dx \]

\[ 3y + y^2 = \sin 2x + C \]

Apply the initial condition now to determine \( C \).

\[ 3(-1) + (-1)^2 = 0 + C \quad \Rightarrow \quad C = -2 \]

As a result,

\[ 3y + y^2 = \sin 2x - 2 \]

\[ y^2 + 3y - (\sin 2x - 2) = 0 \]

\[ y(x) = \frac{-3 \pm \sqrt{9 + 4(\sin 2x - 2)}}{2} \]

We choose the plus sign so that the initial condition is satisfied. Therefore,

\[ y(x) = \frac{-3 + \sqrt{1 + 4 \sin 2x}}{2} \]

Inspecting the solution, we see that the maximum occurs when \( \sin 2x = 1 \), or

\[ 2x = \frac{\pi}{2} + 2n\pi, \quad n = 0, \pm 1, \pm 2, \ldots \]

\[ x = \frac{\pi}{4} + n\pi. \]

At this value of \( x \), the solution is

\[ y \left(\frac{\pi}{4} + n\pi\right) = \frac{-3 + \sqrt{5}}{2} \approx -0.38, \]

which means the maxima occur at approximately \( \left(\frac{\pi}{4} + n\pi, -0.38\right) \) for \( n = 0, \pm 1, \pm 2, \ldots \).
Since the solution is only valid on the part of the curve that passes through $x = 0$ (the $y$-axis), the maximum we're interested in is the $n = 0$ one: $(0.79, -0.38)$. 