Problem 27

Consider the initial value problem

\[ y' = ty(4 - y)/3, \quad y(0) = y_0. \]

(a) Determine how the behavior of the solution as \( t \) increases depends on the initial value \( y_0 \).

(b) Suppose that \( y_0 = 0.5 \). Find the time \( T \) at which the solution first reaches the value 3.98.

Solution

Part (a)

This ODE is separable because it is of the form \( y' = f(t)g(y) \), so it can be solved by separating variables.

\[ \frac{dy}{dt} = \frac{ty(4 - y)}{3} \]

Bring the terms with \( y \) to the left and bring the terms with \( t \) to the right.

\[ \frac{dy}{y(4 - y)} = \frac{t}{3} dt \]

Integrate both sides.

\[ \int \frac{dy}{y(4 - y)} = \int \frac{t}{3} dt \]

\[ \frac{1}{4} \left( \frac{1}{y} + \frac{1}{4 - y} \right) dy = \frac{t^2}{6} + C \]

\[ \frac{1}{4} \int \left( \frac{1}{y} - \frac{1}{y - 4} \right) dy = \frac{t^2}{6} + C \]

Multiply both sides by 4 and evaluate the integral.

\[ \ln |y| - \ln |y - 4| = \frac{2}{3} t^2 + 4C \]

\[ \ln \left| \frac{y}{y - 4} \right| = \frac{2}{3} t^2 + 4C \]

Exponentiate both sides.

\[ \left| \frac{y}{y - 4} \right| = \exp \left( \frac{2t^2}{3} + 4C \right) \]

\[ = e^{4C} e^{2t^2/3} \]

Introduce \( \pm \) on the right side to remove the absolute value sign.

\[ \frac{y}{y - 4} = \pm e^{4C} e^{2t^2/3} \]

Use a new constant \( A \) for \( \pm e^{4C} \).

\[ \frac{y}{y - 4} = Ae^{2t^2/3} \]

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Apply the initial condition \( y(0) = y_0 \) now to determine \( A \).

\[
\frac{y_0}{y_0 - 4} = A
\]

As a result,

\[
\frac{y}{y - 4} = \frac{y_0}{y_0 - 4} e^{2t^2/3}.
\]

Multiply both sides by \( y - 4 \).

\[
y = y \left( \frac{y_0}{y_0 - 4} e^{2t^2/3} - 4 \frac{y_0}{y_0 - 4} e^{2t^2/3} \right)
\]

\[
y \left( 1 - \frac{y_0}{y_0 - 4} e^{2t^2/3} \right) = -4 \frac{y_0}{y_0 - 4} e^{2t^2/3}
\]

Therefore, the solution is

\[
y(t) = \frac{-4 \frac{y_0}{y_0 - 4} e^{2t^2/3}}{1 - \frac{y_0}{y_0 - 4} e^{2t^2/3}}
\]

\[
= \frac{4y_0 e^{2t^2/3}}{4 - y_0 + y_0 e^{2t^2/3}}
\]

\[
= \frac{4y_0 e^{2t^2/3}}{4 + y_0 (e^{2t^2/3} - 1)}.
\]

Figure 1: This figure shows the direction field for the ODE. In red are possible solutions to the ODE, depending what \( y_0 \) is. If \( y_0 > 0 \), then the solution converges to \( y = 4 \). Otherwise, the solution diverges to \(-\infty\).
Part (b)

Set \( y_0 = 0.5 \) and \( y = 3.98 \) and solve the equation for \( t \).

\[
3.98 = \frac{4(0.5)e^{2t^2/3}}{4 + 0.5(e^{2t^2/3} - 1)}
\]

\[
3.98 = \frac{4e^{2t^2/3}}{8 + e^{2t^2/3} - 1}
\]

Multiply both sides by \( 8 + e^{2t^2/3} - 1 \).

\[
31.84 + 3.98e^{2t^2/3} - 3.98 = 4e^{2t^2/3}
\]

\[
27.86 = 0.02e^{2t^2/3}
\]

\[
1393 = e^{2t^2/3}
\]

\[
\ln 1393 = \frac{2t^2}{3}
\]

\[
t^2 = \frac{3}{2} \ln 1393
\]

Therefore,

\[
t = \sqrt{\frac{3}{2} \ln 1393} \approx 3.30.
\]