Problem 29

Solve the equation

$$\frac{dy}{dx} = \frac{ay+b}{cy+d},$$

where a, b, c, and d are constants.

Solution

Make the change of variables,

$$\begin{aligned} u &= ay+b & \quad\rightarrow\quad & y = \frac{u-b}{a} \\ \frac{du}{dx} &= a\frac{dy}{dx} & \quad\rightarrow\quad & \frac{dy}{dx} = \frac{1}{a}\frac{du}{dx}. \end{aligned}$$

As a result, the ODE becomes

$$\begin{aligned} \frac{1}{a}\frac{du}{dx} &= \frac{u}{c\left(\frac{u-b}{a}\right)+d} \\ &= \frac{au}{c(u-b)+ad}. \end{aligned}$$

Multiply both sides by a.

$$\frac{du}{dx} = \frac{a^2u}{c(u-b)+ad}$$

Separate variables by bringing the terms with u to the left and bringing the terms with x to the right.

$$\frac{c(u-b)+ad}{u}\,du = a^2\,dx$$

Integrate both sides.

$$\int \frac{cu+(ad-bc)}{u}\,du = \int a^2\,dx$$

$$\int \left(c + \frac{ad-bc}{u}\right) du = \int a^2\,dx$$

$$cu + (ad-bc)\ln|u| = a^2x + E$$

Therefore, an implicit solution to the ODE is

$$c(ay+b) + (ad-bc)\ln|ay+b| = a^2x + E,$$

where E is a constant of integration.