Problem 30

Consider the equation

$$\frac{dy}{dx} = \frac{y - 4x}{x - y}. \tag{i}$$

(a) Show that Eq. (i) can be rewritten as

$$\frac{dy}{dx} = \frac{(y/x) - 4}{1 - (y/x)}; \tag{ii}$$

thus Eq. (i) is homogeneous.

(b) Introduce a new dependent variable v so that $v = y/x$, or $y = xv(x)$. Express dy/dx in terms of x, v, and dv/dx.

(c) Replace y and dy/dx in Eq. (ii) by the expressions from part (b) that involve v and dv/dx. Show that the resulting differential equation is

$$v + x\frac{dv}{dx} = \frac{v - 4}{1 - v},$$

or

$$x\frac{dv}{dx} = \frac{v^2 - 4}{1 - v}. \tag{iii}$$

Observe that Eq. (iii) is separable.

(d) Solve Eq. (iii), obtaining v implicitly in terms of x.

(e) Find the solution of Eq. (i) by replacing v by y/x in the solution in part (d).

(f) Draw a direction field and some integral curves for Eq. (i). Recall that the right side of Eq. (i) actually depends only on the ratio y/x. This means that integral curves have the same slope at all points on any given straight line through the origin, although the slope changes from one line to another. Therefore, the direction field and the integral curves are symmetric with respect to the origin. Is this symmetry property evident from your plot?

Part (a)

$$\frac{dy}{dx} = \frac{y - 4x}{x - y}$$

Multiply the numerator and denominator by $1/x$.

$$\frac{dy}{dx} = \frac{y - 4x}{x - y} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$

Therefore,

$$\frac{dy}{dx} = \frac{\frac{y}{x} - 4}{1 - \frac{y}{x}}.$$

Part (b)

Make the change of variables,

$$\begin{aligned} v &= \frac{y}{x} & &\rightarrow & y &= xv \\ \frac{dv}{dx} &= \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} & &\rightarrow & \frac{dy}{dx} &= x\frac{dv}{dx} + \frac{y}{x} = x\frac{dv}{dx} + v. \end{aligned}$$

Part (c)

As a result, the ODE becomes

$$\frac{dy}{dx} = \frac{\frac{y}{x} - 4}{1 - \frac{y}{x}} \quad \rightarrow \quad x\frac{dv}{dx} + v = \frac{v-4}{1-v}.$$

Bring v to the right side and simplify.

$$\begin{aligned} x\frac{dv}{dx} &= \frac{v-4}{1-v} - v \\ &= \frac{v - 4 - v(1-v)}{1-v} \\ &= \frac{v^2 - 4}{1-v} \end{aligned}$$

Part (d)

Separate variables by bringing the terms with v to the left and bringing the terms with x to the right.

$$\frac{1-v}{v^2-4}\,dv = \frac{dx}{x}$$

Integrate both sides.

$$\int \frac{1-v}{(v+2)(v-2)}\,dv = \int \frac{dx}{x}$$

$$\int \left(-\frac{3}{4(v+2)} - \frac{1}{4(v-2)}\right) dv = \int \frac{dx}{x}$$

Therefore,

$$-\frac{3}{4}\ln|v+2| - \frac{1}{4}\ln|v-2| = \ln|x| + C.$$

Part (e)

Replace v with y/x.

$$-\frac{3}{4}\ln\left|\frac{y}{x} + 2\right| - \frac{1}{4}\ln\left|\frac{y}{x} - 2\right| = \ln|x| + C.$$

This is an implicit solution to Eq. (i).

Part (f)

The direction field is a two-dimensional vector field that shows what the direction of the solution is at every point in a region. Every solution to the differential equation is a curve drawn such that the direction field vectors are tangent to it at every point.

$$\langle dx, dy \rangle = \left\langle 1, \frac{dy}{dx} \right\rangle dx = \langle 1, \frac{y-4x}{x-y} \rangle \, dx$$

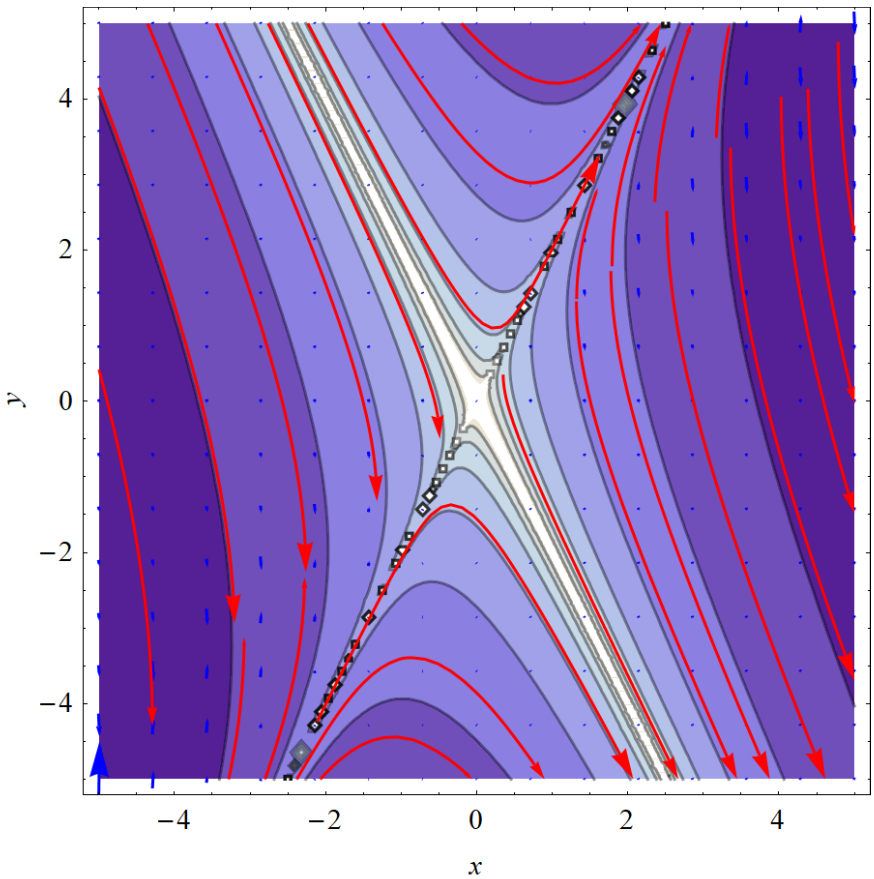

Figure 1: In blue are the direction field vectors and in red are possible solutions to the differential equation, depending what the initial condition is. This plot does have symmetry with respect to the origin.