Problem 3

A tank originally contains 100 gal of fresh water. Then water containing $\frac{1}{2}$ lb of salt per gallon is poured into the tank at a rate of 2 gal/min, and the mixture is allowed to leave at the same rate. After 10 min the process is stopped, and fresh water is poured into the tank at a rate of 2 gal/min, with the mixture again leaving at the same rate. Find the amount of salt in the tank at the end of an additional 10 min.

Solution

Let t represent the time in minutes, let $V = V(t)$ represent the volume in gallons, and let $m = m(t)$ represent the mass of salt in pounds. The tank initially contains 100 gal of water and no salt.

$$V(0) = 100 \text{ gal}$$
$$m(0) = 0 \text{ lb}$$

According to the law of conservation of mass, mass is neither created nor destroyed. If solution flows into a tank at some rate, then it must flow out at the same rate; otherwise, it will accumulate in the tank.

$$\text{rate of accumulation} = \text{rate flowing in} - \text{rate flowing out}$$

Apply this law to the volume, noting that dV/dt is the rate that volume increases with respect to time.

$$\begin{aligned}\frac{dV}{dt} &= 2\,\frac{\text{gal}}{\text{min}} - 2\,\frac{\text{gal}}{\text{min}} \\ &= 0\end{aligned}$$

Integrate both sides with respect to t.

$$V(t) = C_1$$

Use the initial condition for V to determine C_1.

$$V(0) = C_1 = 100 \text{ gal}$$

So the volume is

$$V(t) = 100 \text{ gal}.$$

This is valid for all $t \geq 0$. Now apply the law to the mass, noting that dm/dt is the rate that the mass of salt increases with respect to time. To obtain the rate of mass flow, multiply the concentration by the volume flow rate. Assuming the solution is well-stirred, the concentration flowing out is $m(t)/V(t)$.

$$\begin{aligned}\frac{dm}{dt} &= \left(2\,\frac{\text{gal}}{\text{min}}\right)\left(\frac{1}{2}\,\frac{\text{lb}}{\text{gal}}\right) - \left(2\,\frac{\text{gal}}{\text{min}}\right)\left(\frac{m(t)}{V(t)}\right) \\ &= 1 - \frac{m}{50}\end{aligned}$$

Bring $m/50$ to the left side.

$$\frac{dm}{dt} + \frac{m}{50} = 1$$

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^t \frac{ds}{50}\right) = e^{t/50}$$

Proceed with the multiplication.

$$e^{t/50}\frac{dm}{dt} + \frac{1}{50}e^{t/50}m = e^{t/50}$$

The left side can be written as $d/dt(Im)$ by the product rule.

$$\frac{d}{dt}(e^{t/50}m) = e^{t/50}$$

Integrate both sides with respect to t.

$$\begin{aligned} e^{t/50}m &= \int^t e^{s/50}\,ds + C_2 \\ &= 50e^{t/50} + C_2 \end{aligned}$$

Divide both sides by $e^{t/50}$.

$$m(t) = 50 + C_2 e^{-t/50}$$

Use the initial condition for m to determine C_2.

$$m(0) = 50 + C_2 = 0 \quad \rightarrow \quad C_2 = -50$$

So then the mass of salt is

$$\begin{aligned} m(t) &= 50 - 50e^{-t/50} \\ &= 50(1 - e^{-t/50})\text{ lb}, \quad 0 \le t \le 10 \end{aligned}$$

for the first ten minutes. At the ten-minute mark, the mass of salt in the tank is

$$m(10) = 50(1 - e^{-10/50})\text{ lb} \approx 9.06\text{ lb}.$$

This serves as the initial condition for the second part of the process in which fresh water is poured into the tank. For $t \ge 10$, the law of mass conservation yields

$$\begin{aligned} \frac{dm}{dt} &= \left(2\,\frac{\text{gal}}{\text{min}}\right)\left(0\,\frac{\text{lb}}{\text{gal}}\right) - \left(2\,\frac{\text{gal}}{\text{min}}\right)\left(\frac{m(t)}{V(t)}\right) \\ &= -\frac{m}{50} \end{aligned}$$

Solve this ODE by separating variables.

$$\frac{dm}{m} = -\frac{dt}{50}$$

Integrate both sides.

$$\begin{aligned} \ln|m| &= -\frac{t}{50} + \frac{10}{50} + C_3 \\ &= -\frac{t-10}{50} + C_3 \end{aligned}$$

Exponentiate both sides.

$$\begin{aligned}|m| &= \exp\left(-\frac{t-10}{50}+C_3\right)\\ &= e^{C_3}\exp\left(-\frac{t-10}{50}\right)\end{aligned}$$

Introduce $\pm$ on the right side to remove the absolute value sign.

$$m(t) = \pm e^{C_3}\exp\left(-\frac{t-10}{50}\right)$$

Use a new integration constant A for $\pm e^{C_3}$.

$$m(t) = A\exp\left(-\frac{t-10}{50}\right)$$

Apply the initial condition for the second part of the process to determine A.

$$m(10) = A = 50(1-e^{-10/50})$$

After ten minutes, then, the mass of salt in the tank is

$$m(t) = 50(1-e^{-10/50})\exp\left(-\frac{t-10}{50}\right)\text{ lb},\quad t\geq 10.$$

Therefore,

$$m(t) = \begin{cases} 50(1-e^{-t/50}) & \text{if } 0\leq t\leq 10\\ 50(1-e^{-10/50})\exp\left(-\frac{t-10}{50}\right) & \text{if } t\geq 10\end{cases},$$

and the mass of salt at 20 minutes is

$$m(20) = 50(1-e^{-10/50})\exp\left(-\frac{20-10}{50}\right)\text{ lb}\approx 7.42\text{ lb}.$$