Problem 7

Suppose that a sum $S_0$ is invested at an annual rate of return $r$ compounded continuously.

(a) Find the time $T$ required for the original sum to double in value as a function of $r$.

(b) Determine $T$ if $r = 7\%$.

(c) Find the return rate that must be achieved if the initial investment is to double in 8 years.

Solution

If the compounding is continuous, then the rate $dS/dt$ that the sum of money grows with respect to time is equal to the annual interest $r$ times $S$.

$$
\frac{dS}{dt} = rS
$$

Divide both sides by $S$.

$$
\frac{dS}{S} = r dt
$$

The left side can be written as the derivative of a logarithm.

$$
\frac{d}{dt} \ln S = r
data=
it
\ln S = rt + C
$$

Integrate both sides with respect to $t$.

Exponentiate both sides.

$$
S(t) = e^{rt+C}
$$

$$
= e^C e^{rt}
$$

Use a new constant $A$ for $e^C$.

$$
S(t) = Ae^{rt}
$$

Assuming that the initial investment is $S(0) = S_0$, the constant $A$ evaluates to

$$
S(0) = A = S_0.
$$

So then the amount of money available at any time $t$ (in years) is

$$
S(t) = S_0 e^{rt}.
$$
Part (a)

Set $S = 2S_0$ and solve for $t = T$ to determine how long it takes for the initial investment to double in value.

$$2S_0 = S_0 e^{rT}$$

$$e^{rT} = 2$$

$$\ln e^{rT} = \ln 2$$

$$rT = \ln 2$$

Therefore,

$$T = \frac{\ln 2}{r}.$$

Part (b)

If $r = 7\% = 0.07$, then

$$T = \frac{\ln 2}{0.07} \approx 9.90 \text{ years}.$$

Part (c)

Set $T = 8$ and solve for $r$ in the result of part (a).

$$8 = \frac{\ln 2}{r}$$

$$r = \frac{\ln 2}{8} \approx 0.0866 = 8.66\%$$