Problem 8

A young person with no initial capital invests $k$ dollars per year at an annual rate of return $r$. Assume that investments are made continuously and that the return is compounded continuously.

(a) Determine the sum $S(t)$ accumulated at any time $t$.

(b) If $r = 7.5\%$, determine $k$ so that $1$ million will be available for retirement in 40 years.

(c) If $k = 2000$/year, determine the return rate $r$ that must be obtained to have $1$ million available in 40 years.

Solution

Part (a)

The young man’s capital $S(t)$ grows in time due to two factors, the compound interest and his continuous investments. The rate of growth for compounding is $rS$, and the rate of growth due to the continuous investments is $k$.

$$\frac{dS}{dt} = rS + k$$

Bring $rS$ to the left side.

$$\frac{dS}{dt} - rS = k$$

This is a linear first-order inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor $I$.

$$I = \exp\left(\int (-r) \, ds\right) = e^{-rt}$$

Proceed with the multiplication.

$$e^{-rt} \frac{dS}{dt} - re^{-rt} S = ke^{-rt}$$

The left side can be written as $d/dt(IS)$ by the product rule.

$$\frac{d}{dt}(e^{-rt} S) = ke^{-rt}$$

Integrate both sides with respect to $t$.

$$e^{-rt} S = -\frac{k}{r} e^{-rt} + C$$

Multiply both sides by $e^{rt}$.

$$S(t) = -\frac{k}{r} + Ce^{rt}$$

Apply the initial condition $S(0) = 0$ to determine $C$.

$$S(0) = -\frac{k}{r} + C = 0 \quad \Rightarrow \quad C = \frac{k}{r}$$
Therefore, the young man’s capital after \( t \) years is

\[
S(t) = -\frac{k}{r} + \frac{k}{r} e^{rt} \nonumber
\]

\[
= \frac{k}{r} (e^{rt} - 1) \nonumber.
\]

**Part (b)**

Set \( r = 0.075 \), \( S(t) = 1000 \, 000 \), \( t = 40 \), and solve the resulting equation for \( k \).

\[
1000 \, 000 = \frac{k}{0.075} (e^{0.075 \cdot 40} - 1) \nonumber
\]

\[
k = \frac{0.075 \cdot 1000 \, 000}{e^{0.075 \cdot 40} - 1} \approx 3930 \text{ dollars/ year} \nonumber.
\]

**Part (c)**

Set \( k = 2000 \), \( S(t) = 1000 \, 000 \), \( t = 40 \), and solve the resulting equation for \( r \).

\[
1000 \, 000 = \frac{2000}{r} (e^{40r} - 1) \nonumber
\]

\[
500r = e^{40r} - 1 \nonumber
\]

Plot \( y = 500r \) and \( y = e^{40r} - 1 \) on the same axis and find where the two curves intersect.

We see that \( r \approx 0.098 = 9.8\% \). This is how high the annual interest rate has to be to have a million dollars in 40 years by only continuously investing $2000 annually.