Problem 10

A home buyer can afford to spend no more than $1500/month on mortgage payments. Suppose that the interest rate is 6%, that interest is compounded continuously, and that payments are also made continuously.

(a) Determine the maximum amount that this buyer can afford to borrow on a 20-year mortgage; on a 30-year mortgage.

(b) Determine the total interest paid during the term of the mortgage in each of the cases in part (a).

Solution

The amount of money $S(t)$ that the buyer has to pay changes in time due to two factors, the compound interest and his continuous payments. The rate of growth for compounding is $rS$, and the rate of decay due to the continuous payments is $k$.

\[
\frac{dS}{dt} = rS - k
\]

Bring $rS$ to the left side.

\[
\frac{dS}{dt} - rS = -k
\]

This is a linear first-order inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor $I$.

\[
I = \exp\left(\int (-r) \, ds\right) = e^{-rt}
\]

Proceed with the multiplication.

\[
e^{-rt} \frac{dS}{dt} - re^{-rt} S = -ke^{-rt}
\]

The left side can be written as $d/dt(IS)$ by the product rule.

\[
\frac{d}{dt}(e^{-rt} S) = -ke^{-rt}
\]

Integrate both sides with respect to $t$.

\[
e^{-rt} S = \frac{k}{r} e^{-rt} + C
\]

Multiply both sides by $e^{rt}$.

\[
S(t) = \frac{k}{r} + Ce^{rt}
\]

Apply the initial condition $S(0) = S_0$ to determine $C$.

\[
S(0) = \frac{k}{r} + C = S_0 \quad \rightarrow \quad C = S_0 - \frac{k}{r}
\]
Therefore, the amount of money the buyer has to pay after $t$ years is

$$S(t) = \frac{k}{r} + \left( S_0 - \frac{k}{r} \right) e^{rt} \quad = \frac{k}{r}(1 - e^{rt}) + S_0 e^{rt}.$$ 

Set $k = 1500 \times 12 = 18000$ dollars/year and $r = 6\% = 0.06$.

$$S(t) = \frac{18000}{0.06}(1 - e^{0.06t}) + S_0 e^{0.06t}$$

For a 20-year mortgage, $S(t) = 0$ at $t = 20$.

$$0 = \frac{18000}{0.06}(1 - e^{0.06 \cdot 20}) + S_0 e^{0.06 \cdot 20} \quad \rightarrow \quad S_0 \approx 209641.74$$

For a 30-year mortgage, $S(t) = 0$ at $t = 30$.

$$0 = \frac{18000}{0.06}(1 - e^{0.06 \cdot 30}) + S_0 e^{0.06 \cdot 30} \quad \rightarrow \quad S_0 \approx 250410.33$$

These values of $S_0$ are how much the buyer can afford to borrow initially. For the 20-year mortgage, he pays a total of $18000 \times 20 = $360000, so the total interest he pays is

$$360000 - 209641.74 \approx 150358.26.$$ 

For the 30-year mortgage, he pays a total of $18000 \times 30 = $540000, so the total interest he pays is

$$540000 - 250410.33 \approx 289589.67.$$